QUASILIMITING BEHAVIOR FOR ONE-DIMENSIONAL DIFFUSIONS WITH KILLING

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ABSTRACT. This paper extends and clarifies results of Steinsaltz and Evans [47], which found conditions for convergence of a killed one-dimensional diffusion conditioned on survival, to a quasistationary distribution whose density is given by the top eigenfunction of the generator. Convergence occurs when the limit of the killing at ∞ differs from the negative top of the spectrum of the generator. When the killing at ∞ is *larger* than the negative top of the spectrum, then the eigenfunction is integrable. When the killing at ∞ is *smaller*, the eigenfunction is integrable only when the unkilled process is recurrent; otherwise, the process conditioned on survival converges to 0 density on any bounded interval.

1. Introduction

1.1. **Background and history.**

Killed Markov processes are central objects in probability theory. When considering their asymptotic behavior there are two distinct approaches, which may be thought of as two different ways of ordering the different limits:

- We condition on the process never being killed. That is, we look at the distribution of $\{X_t : t \in [0,s]\}$ for fixed s conditioned on $\tau_\partial > T$ (where τ_∂ is the killing time), in the limit as $T \to \infty$; we may then take a second limit $s \to \infty$ to define the process on $[0,\infty)$. Following Doob [17], this is called an **h-process** or an **h-transform** of the original Markov process, and it generally produces a new Markov process. A well-known example of this is the 3-dimensional Bessel process, which may be derived from the one-dimensional Brownian motion, conditioned never to hit 0 [50, Section 6.6]. This is intimately connected to questions about the Martin boundary.
- We condition on the process not having been killed up to the time in question. That is, we look at the distribution of X_t conditioned on $\tau_\partial > t$, in the limit as $t \to \infty$. The collection of distributions for different times t are not consistent, and so cannot be analyzed directly with Markov-process techniques, but are more amenable to an analytic semigroup approach. We generally expect the distribution to converge to a limit — sometimes called the **Yaglom limit**, after the seminal work of Yaglom [56] on branching Markov processes conditioned on long survival — which will be **quasistationary**, in the sense that when started in this distribution the process will remain in a multiple of the same distribution for all times.

The extensive mathematical development and wide-ranging applications in this area — a bibliography of papers on quasistationary distributions and Yaglom limits compiled and periodically updated by Phil Pollett [42] lists 375 entries through 2005 — permit us to mention only a smattering of the vast array of applications of killed Markov processes to biology

2000 *Mathematics Subject Classification.* Primary 58J05; Secondary 35J10, 35P05, 47F05.
Key words and phrases. diffusion,killing, quasistationary distribution, spectral theory, Sturm-Liouville, operator.

[44, 23, 22, 9], demography [46, 29, 30], medicine [34, 57], and statistics [1, 2, 15]. Particularly in the demographic and medical contexts, where killed Markov processes suggest themselves as models for populations undergoing culling by mortality or other processes, Yaglom limits, while rarely mentioned explicitly in the applied literature, correspond naturally to the observable distribution of survivors.

The central concerns of this theory are to describe, for a given class of sub-Markov processes, the quasistationary distributions (if any), and to describe the convergence (or not) of the process conditioned on survival to one of these quasistationary distributions. A significant part of the literature focuses on discrete state spaces, commonly birth-death processes (or with some more flexible localization of the transitions), with killing only on the boundary. [18] gave one of the most general accounts of the existence and convergence to quasistationary distributions for discrete processes of this kind. One unusual contribution, outside of these categories, is [21], which proves convergence to quasistationarity for fairly arbitrary discrete Markov chains with general killing, by imposing a stringent Lyapunov-like drift condition. [28] generalize in a different direction, finding existence and vague-convergence conditions for discrete-time Markov chains on general metric spaces.

Killed birth-death processes naturally generalise in the continuous-space context to killed diffusions, which have received rather less attention. The existence of eigenfunctions for the generator of a one-dimensional diffusions is simplified in the continuous setting, as we may rely upon standard theory of ordinary differential equations. Showing that these eigenfunctions are integrable (hence represent the densities of distributions) and quasistationary is more involved, though, and showing Yaglom convergence to the minimal quasistationary distribution becomes technically challenging, particularly when the state space is an unbounded interval. The basis for all later work on Yaglom convergence of diffusions was laid by Mandl [33], who used standard results from Sturm-Liouville theory and the spectral theorem for self-adjoint operators to prove vague convergence (that is, convergence of the distribution of the process conditioned on being in a compact set), and uniform convergence under an assumption of strong inward drift. These results have been substantially extended by a shifting coalition of researchers who have produced papers [12, 35, 9], which elucidate the conditions under which Yaglom convergence occurs, and distinguish in [36] the R-positive situation from the R-null — essentially, exponential-rate decline of probabilities distinguished from declines that are asymptotically not exactly exponential — in terms of the eigenfunctions.

One important constraint in nearly all of this work — as well as Pinsky's results in [39], for diffusions on a compact domain with gradient-type drift — is the assumption that killing occurs only at the boundary. Not only is this an unnatural assumption from the perspective of many of the applications, particularly the demographic applications, as discussed in [47, 46], we argue here that it obscures the fundamental links among the spectrum, the killing rate out at infinity, the recurrence-transience dichotomy, and Yaglom convergence. (An exception which proves the rule is the biological application of internally killed diffusions [25], which makes no reference to any of the literature on killed diffusions, and cites only [43] for quasistationary distributions of discrete chains.) In [47] it was shown that a key criterion for Yaglom convergence is whether the limit of the killing rate out at infinity is greater than or less than the bottom of the diffusion operator spectrum λ_0^κ, which may be interpreted as the rate of being killed or escaping from a compact interval around the origin. Here we substantially complete that program of [47], providing testable criteria for distinguishing convergent processes from divergent processes that escape to infinity when conditioned on long survival, in all cases when the limit killing rate at infinity exists and is not exactly equal to λ_0^κ.

1.2. **Heuristics.** We begin by observing that general spectral theory — summarized here in Lemma 2.1 — tells us that the bottom of the spectrum λ_0^κ gives the exponential rate of decay of the distribution of X_t restricted to a compact interval. What needs to be addressed, then, is the question of whether the portion of the surviving mass within a compact interval dominates the total surviving mass. There are two ways of addressing this question. One is in terms of the spectrum of the $\mathfrak{L}^2$ generator. Consider the case when λ_0^κ is an isolated eigenvalue, as when the diffusion takes place on a compact interval with two regular boundaries, so that the resolvent is compact; but also in cases which intuitively seem well-approximated by a compact process, as when there is strong drift pulling the process in from ∞, and when there is strong killing out toward ∞. Then we expect the same methods that work in finite dimensions, for powers of positive symmetric matrices, to work here as well. One catch is that the $\mathfrak{L}^2$ convergence need not tell us about the convergence of the conditioned density, which is an $\mathfrak{L}^1$ property; indeed, it is easy to see (*cf.* Proposition 2.3 of [47]) that the process always escapes to ∞ if $\int_0^\infty \varphi\lambda_0^\kappa, x)d\Gamma(x)$ is not finite, where Γ is the speed measure and $\varphi(\lambda, \cdot)$ is the eigenfunction of the generator, defined as the solution to an ordinary differential equation in section 3.2. We show, in Lemma 4.2 that these conditions suffice: That is, whenever λ_0^κ is an isolated eigenvalue, and the corresponding eigenfunction is also integrable, then we have convergence to the quasistationary distribution given by the density $\varphi(\lambda_0^\kappa, \cdot)/\int\varphi(\lambda_0^\kappa, x)d\Gamma(x)$.

What about the case when λ_0^κ is not an isolated eigenvalue? This corresponds to the R-null and R-transient cases in Tweedie's theory [49, 48], where the decline of the transition kernel is not exactly exponential with rate $-\lambda_0^\kappa$, but slightly faster, in the sense that $e^{\lambda_0^\kappa t}p(t, x, y) \to 0$. When λ_0^κ is greater than the limit of killing at ∞ it turns out that the convergence lines up precisely with the standard recurrence/transience dichotomy for the unkilled process. (Another way of putting this is to say that when the R-recurrence or R-transience does not conform to the properties of the unkilled process, this must be reflected in the equality of λ_0^κ and $\lim_{x\to\infty}\kappa(x)$.) As summarized in Lemma 3.3, if $\lambda_0^\kappa > \limsup_{x\to\infty}\kappa(x)$ — if the distribution declines on compact sets at a faster exponential rate than would keep pace with the killing out toward ∞ — this implies an exponential decline on compact sets even in the absence of internal killing (by which we mean killing by κ, rather than instantaneous killing at the boundary). This means either that the scale is finite, so that the unkilled process runs off toward ∞, and the distribution on compact sets would decline exponentially in the absence of any killing at all — the R-transient case — or the scale is infinite with finite speed, which means that the process keeps returning to 0, at long intervals, and the killing rate λ_0 corresponds to real killing at 0, not escape, which is the R-null case. In the R-transient case the conditioned process escapes to infinity. In the R-null case the conditioned process converges to the quasistationary distribution.

1.3. **Main results.** The core of this work is the identification of the asymptotic behavior of killed diffusions in terms of the eigenvalues of the generator and the finiteness of the speed and scale measures. We move beyond earlier work in eliminating the supplementary bounds on the drift and killing terms, and in providing easily testable criteria for determining whether the conditioned process converges to a quasistationary distribution for all cases in which the bottom of the spectrum λ_0^κ does not coincide with the limit of the killing rate at ∞.

We begin by summarising the most important results. These results presuppose general assumptions and restrictions on the processes involved, which will be formulated fully in section 2. The quasistationary distribution will be defined in terms of its density $\varphi(\lambda_0^\kappa, \cdot)$ with respect to Γ, where φ is the eigenfunction of the generator, defined as the solution to an

ordinary differential equation with appropriate boundary condition, stated formally in section 3.2.

(i) There is always convergence to the quasistationary distribution on compact sets, stated formally as Theorem 3.15.

(ii) If $\lambda_0^\kappa > \limsup_{x\to\infty} \kappa(x)$ or $\lambda_0^\kappa < \liminf_{x\to\infty} \kappa(x)$, then the conditioned process either converges to the quasistationary distribution with density $\varphi(\lambda_0^\kappa, \cdot) \left(\int_0^\infty \varphi(\lambda_0^\kappa, y) d\Gamma(y)\right)^{-1}$ with respect to Γ, or escapes to ∞. This behavior is independent of the initial distribution, provided only that it is compactly supported. (For explanation of the terminology, see section 2.3.) This is Theorem 3.3 of [47], but it is restated here as Theorem 2.6 in a slightly stronger form, as several restrictions have been removed.

(iii) If $\lambda_0^\kappa < \limsup_{x\to\infty} \kappa(x)$ then the conditioned process converges to the quasistationary distribution. This is stated as Theorem 4.1. Note that this includes the (somewhat unintuitive) fact that a bound on the $\mathfrak{L}^2$ spectrum implies that $\int_0^\infty u_{\lambda_0^\kappa}(z)\gamma(z)dz < \infty$, which is a fact about the $\mathfrak{L}^1$ spectrum.

(iv) If $K := \lim_{x\to\infty} \kappa(x)$ exists and $K < \lambda_0^\kappa$, then the behavior of the conditioned process depends on the transience or recurrence of the unkilled process. If the unkilled process is transient — that is, if $\int_0^\infty \gamma(x)^{-1}dx < \infty$ — then the conditioned process escapes to ∞. If the unkilled process is recurrent — that is, if $\int_0^\infty \gamma(x)^{-1}dx = \infty$ — then the conditioned process converges to the quasistationary distribution. These results are stated in Theorems 4.9 and 4.7.

These extend similar results from [47]. The differences are as follows:

- The boundary at ∞ was assumed in the earlier work to be natural. Here, it may be an entrance boundary as well.
- In [47] there were conditions imposed that required the drift and killing not to grow too quickly, or be too irregular. Here there is no constraint on the killing other than local boundedness, and no constraint on the drift other than that which implies that ∞ is inaccessible.
- The earlier work imposed an unintuitive condition on the initial distribution that could be hard to check. Here the results all presuppose only a compactly supported initial distribution.
- The most substantial advance: The case $K < \lambda_0^\kappa$ is now shown to be split by the standard recurrence-transience dichotomy, which tells us whether the conditioned process converges or escapes. In [47] this was left open.

2. Assumptions, Definitions and Previous Results

2.1. **Analytic terminology.** In general a Sturm-Liouville operator is any formal differential operator of the form $\tau = \tau_{p,q,V} = -\frac{1}{2p}\frac{d}{dx}q\frac{d}{dx} + V$, where $p, q : (a, b) \to (0, \infty)$ and $V : (a, b) \to \mathbb{R}$ are sufficiently well behaved functions. In this work we consider only operators where $p = q = \gamma$, $V = \kappa \geq 0$ and $a = 0$, $b = \infty$. Moreover, we always assume in this chapter that $\gamma(x) = e^{2\int_0^x b(s)\,ds}$ for some $b \in \mathfrak{L}^1_{loc}([0, \infty)) \cap C((0, \infty))$ and $0 \leq \kappa \in C\big([0, \infty)\big)$. These conditions are not entirely necessary, but this constraint still admits a large class of one-dimensional diffusions. (However, see [9] for a natural application to biology which requires b to be singular at 0.) Concerning the assumptions on b we could replace the condition $b \in \mathfrak{L}^1_{loc}([0, \infty))$ by the condition that $\int_0^1 e^{-\int_c^x 2b(s)\,ds}\,dx < \infty$ for some $c \in (0, \infty)$, and $\int_0^1 e^{\int_c^x 2b(s)\,ds}\,dx < \infty$, which is equivalent to saying that the boundary point 0 is regular in

the sense of Feller and also in the sense of Weyl. The formal differential operator $L^\kappa = -\frac{1}{2\gamma}\frac{d}{dx}\gamma\frac{d}{dx}+\kappa$ gives rise to a closable densely defined quadratic form $\tilde q^\kappa$ in $\mathfrak{L}^2$ by

$$\varphi \mapsto \tilde q^{\kappa,\alpha}(\varphi) = \begin{cases} \alpha\phi(0)^2 + \frac12\int_0^\infty |\varphi'(y)|^2\gamma(y)\,dy + \int_0^\infty \kappa(y)|\varphi(y)|^2\,\gamma(y)\,dy & \text{if } \alpha<\infty,\\ \frac12\int_0^\infty |\varphi'(y)|^2\gamma(y)\,dy + \int_0^\infty \kappa(y)|\varphi(y)|^2\,\gamma(y)\,dy & \text{if } \alpha=\infty. \end{cases} \tag{1}$$

for any $\varphi\in\mathcal{D}_\alpha$, where $\alpha = p_0/(1-p_0)\in[0,\infty]$, and the domain $\mathcal{D}_\alpha$ is

$$\mathcal{D}_\alpha := \big\{\varphi\in\mathfrak{L}^2 \mid \varphi,\,\gamma\varphi' \text{ absolutely continuous in } (0,\infty),\ \tilde q^\kappa(\varphi)<\infty,\ (1-p_0)\phi(0)=\frac12 p_0\phi'(0)\big\}$$

The closure of this quadratic form will be denoted by $q^{\kappa,\alpha}$. In the sequel Γ will denote the measure $\gamma(y)\,dy$. (This is identical to the "speed measure" of the diffusion.) Where there is no need to emphasise the particular boundary condition at 0, the parameter α will be suppressed.

In this paper we will consistently use Γ as a reference measure instead of the Lebesgue measure, which is different from the convention adopted in [47]. Recall that the speed measure of a one-dimensional diffusion is also the reversing measure, with respect to which the generator is symmetric. Unless otherwise indicated, we will always use the bare notation $\mathfrak{L}^2$ to mean $\mathfrak{L}^2\big((0,\infty),\Gamma\big)$, and for $f,g\in\mathfrak{L}^2$ we have the inner product

$$\langle f,g\rangle = \int_0^\infty f(x)g(x)\gamma(x)dx. \tag{2}$$

To the quadratic form $q^{\kappa,\alpha}$ there corresponds a uniquely defined positive selfadjoint operator $L^{\kappa,\alpha}$. When $\alpha=0$ this is the so-called Friedrichs extension. It is easy to see that the action of the operator $L^{\kappa,\alpha}$ is given by

$$L^{\kappa,\alpha}\varphi(x) = -\frac12\varphi''(x) - b(x)\varphi'(x) + \kappa(x)\varphi(x).$$

The bottom of the spectrum of L^κ will be denoted by λ_0^κ. The spectrum of the selfadjoint operator L^κ is written $\Sigma(L^\kappa)$. Where there is no danger of confusion, the corresponding objects with $\kappa\equiv 0$ will also be denoted by q, L and λ_0 instead of q^0, L^0 and λ_0^0, respectively (or $q^{0,\alpha}$, $L^{0,\alpha}$, and $\lambda_0^{0,\alpha}$. Since L^κ and L are selfadjoint operators the spectral theorem implies the existence of spectral resolutions $(E^\kappa_\lambda)_{\lambda\in[\lambda_0^\kappa,\infty)}$ and $(E_\lambda)_{\lambda\in[\lambda_0,\infty)}$, respectively. For the basic facts concerning spectral theory of selfadjoint operators the reader should consult [53].

The spectral theorem for self-adjoint operators allows us to define functions $f(L^\kappa)$ of the operator. For every Borel-measurable function $f:\mathbb{R}\to\mathbb{R}$ the operator $f(L^\kappa)$ is defined via

$$\mathcal{D}(f(L^\kappa)) = \Big\{u\in\mathfrak{L}^2 \mid \int_{\Sigma(L^\kappa)} |f(\lambda)|^2\,d\|E^\kappa u\|^2(\lambda)<\infty\Big\}, \tag{3}$$

$$f(L^\kappa)u = \int_{\Sigma(L^\kappa)} f(\lambda)\,dE^\kappa(\lambda)u, \tag{4}$$

$$\big\|f(L^\kappa)u\big\|^2 = \int_{\Sigma(L^\kappa)} f(\lambda)^2 d\|E^\kappa u\|^2(\lambda). \tag{5}$$

Observe that for a Borel-measurable function $f:[0,\infty)\to\mathbb{R}$ and $a\ge 0$ we have $Ran(f(L^\kappa))\subset \mathcal{D}((L^\kappa)^a)$ if $[0,\infty)\ni\lambda\mapsto|\lambda^a f(\lambda)|$ is bounded. This implies in particular that the range of e^{-tL^κ} is contained in the domain of all powers of L^κ. Moreover the spectral theorem allows us to clarify further the connection between the quadratic form q^κ and the associated non-negative operator L^κ. Let $\sqrt{L^\kappa}$ denote the unique non-negative square root of L^κ, which is

defined using the spectral theorem. Then we have $\mathcal{D}(q^\kappa) = \mathcal{D}(\sqrt{L^\kappa})$ and for every $f \in \mathcal{D}(L^\kappa)$ we have

$$q^\kappa(f,g) = \big\langle \sqrt{L^\kappa} f, \sqrt{L^\kappa} g \big\rangle. \tag{6}$$

Using the 'elliptic' Harnack inequality and Weyl's spectral theorem it is not difficult to see that

$$\begin{aligned} \lambda_0^\kappa = \max\{\lambda \in \mathbb{R} \,|&\text{there is a positive solution of } (L^\kappa - \lambda)u = 0 \\ &\text{with } u(0) = \frac{1}{1+\alpha}, \quad u'(0) = \frac{\alpha}{1+\alpha}\}. \end{aligned} \tag{7}$$

(This was proved by [33] using slightly different methods.) Equation (7) already suggests that for $0 \le \lambda \le \lambda_0^\kappa$ solutions of $(L^\kappa - \lambda)u = 0$ might have a probabilistic significance.

In the sequel we usually denote by $\varphi(\lambda, \cdot)$ the solution of the eigenvalue equation

$$(L^\kappa - \lambda)\varphi(\lambda, \cdot) = 0, \;\; \varphi(\lambda, 0) = \frac{1}{1+\alpha}, \; \varphi'(\lambda, 0) = \frac{\alpha}{1+\alpha}. \tag{8}$$

It might be important to note that solutions in (7) and (8) are solutions in the sense of the theory of ordinary differential equations. An important issue is whether the solution also belongs to the Hilbert space $\mathfrak{L}^2$, and thus is an eigenfunction in the sense of spectral theory. When we wish to emphasise that certain solution are also eigenfunctions in the sense of spectral theory, we denote them by u_λ.

Crucial to much of our analysis is the fact that the asymptotic behaviour of the semigroup is wholly determined by the spectrum right near the base of the spectral measure, which we show in Lemma 2.1, and then that the base of the spectral measure for any nonnegative function is λ_0^κ, which is Lemma 2.2. For $g \in \mathfrak{L}^2$, define λ_g to be the infimum of the support of the spectral measure of g; that is,

$$\lambda_g := \sup\big\{\lambda \,:\, \|E_\lambda g\| = 0\big\}, \tag{9}$$

and let $\mathcal{A}_\lambda$ be the subspace of $\mathfrak{L}^2$ consisting of functions f such that $\lambda_f \ge \lambda$.

Lemma 2.1. *Given $g \in \mathcal{D}(L^{\kappa,\alpha})$, we have*

$$\begin{aligned} |g(x)| &\le C_\alpha(x) \big\|\sqrt{L^\kappa} g\big\| + C'_\alpha \|g\| \\ &= C_\alpha(x) \left(\int_0^\infty \lambda d\|E^\kappa g\|^2(\lambda)\right)^{1/2} + C'_\alpha \left(\int_0^\infty d\|E^\kappa g\|^2(\lambda)\right)^{1/2}, \end{aligned} \tag{10}$$

where

$$C_\alpha(x) := \begin{cases} \max\left\{\sqrt{\frac{2}{\alpha}}, \big(2\int_0^x \gamma(y)^{-1}dy\big)^{1/2}\right\} & \text{for } \alpha > 0, \\ \big(18\int_0^x \gamma(y)^{-1}dy\big)^{1/2} & \text{for } \alpha = 0, \end{cases} \tag{11}$$

and

$$C'_\alpha := \begin{cases} 0 & \text{if } \alpha > 0 \text{ or } \int_0^\infty \gamma(y)dy = \infty, \\ \big(\int_0^\infty \gamma(y)dy\big)^{-1/2} & \text{if } \alpha = 0 \text{ and } \int_0^\infty \gamma(y)dy < \infty. \end{cases} \tag{12}$$

For any $t > 1/2\lambda_g$,

$$\sup |e^{-tL^\kappa} g(x)| \le \big(C_\alpha(x)\lambda_g + C'_\alpha\big) \|g\| e^{-t\lambda_g}. \tag{13}$$

Proof. Suppose $\alpha \in (0,\infty)$. Since $g \in \mathcal{D}(L^\kappa)$ is differentiable, we have

$$\begin{aligned}
|g(x)| &\le |g(0)| + \int_0^\infty \left|g'(y)\right| \frac{\mathbf{1}_{[0,x]}}{\gamma(y)} \gamma(y) dy \\
&= |g(0)| + \left\langle |g'|, \frac{\mathbf{1}_{[0,x]}}{\gamma} \right\rangle \\
&\le |g(0)| + \left\| \frac{\mathbf{1}_{[0,x]}}{\gamma} \right\| \cdot \|g'\| \qquad \text{(Cauchy-Schwarz inequality)} \\
&\le \left(2|g(0)|^2 + 2 \left\| \frac{\mathbf{1}_{[0,x]}}{\gamma} \right\| \int_0^\infty \left|g'(y)\right|^2 \gamma(y) dy \right)^{1/2} \\
&\le C_\alpha q^{\kappa,\alpha}(g)^{1/2} \\
&= C_\alpha \left\| \sqrt{L^{\kappa,\alpha}} g \right\|
\end{aligned}$$

by (1) and (6). The spectral theorem (5) allows us to represent $\sqrt{L^\kappa} g$ in terms of the spectral resolution, yielding (10).

If $\alpha = \infty$ then $g(0) = 0$, so the corresponding term drops out of the bound.

If $\alpha = 0$, we have the alternative bound

$$\begin{aligned}
|g(x)| &\le |g(0)| + C \left(q^{\kappa,0}(g)\right)^{1/2} \\
|g(x)| &\ge |g(0)| - C \left(q^{\kappa,0}(g)\right)^{1/2},
\end{aligned}$$

where $C = \sqrt{2} \left\| \frac{\mathbf{1}_{[0,x]}}{\gamma} \right\|$. The second bound gives us

$$\|g\|^2 \ge \left(\int_0^\infty \gamma(y) dy \right) \left(\left|g(0)\right|^2 - 2C\left|g(0)\right| \left(q^{\kappa,0}(g)\right)^{1/2} \right),$$

which implies that

$$\left|g(0)\right| \le 2C\sqrt{q^{\kappa,0}(g)} + \|g\|^2 \left(\int_0^\infty \gamma(y) dy \right)^{-1/2}.$$

We combine this with the above calculation to obtain the appropriate version of (10).

For any positive t, we have $g_t := e^{-tL^\kappa} g \in \mathcal{D}(L^\kappa)$, so we may apply (10) to obtain

$$|g_t(x)| \le \left(\int_0^x \gamma(y)^{-1} dy \right)^{1/2} \left\| \sqrt{L^\kappa} e^{-tL^\kappa} g \right\|.$$

Applying again the spectral theorem (5) — now with $f(x) = \sqrt{x} e^{-tx}$ — yields

$$\begin{aligned}
\left\| \sqrt{L^\kappa} e^{-tL^\kappa} g \right\|^2 &= \int_0^\infty \lambda e^{-2t\lambda} d \left\| E^\kappa g \right\|^2 (\lambda) \\
&= \int_{\lambda_g}^\infty \lambda e^{-2t\lambda} d \left\| E^\kappa g \right\|^2 (\lambda) \\
&\le \lambda_g e^{-2t\lambda_g} \|g\|^2,
\end{aligned}$$

since $\lambda e^{-2t\lambda}$ attains its maximum at $\lambda = \frac{1}{2t}$. Similarly,

$$\left\| e^{-tL^\kappa} g \right\|^2 = \int_{\lambda_g}^\infty e^{-2t\lambda} d \left\| E^\kappa g \right\|^2 (\lambda) \le e^{-2t\lambda_g} \|g\|^2.$$

□

Lemma 2.2. *For any non-negative measurable function $f \in \mathfrak{L}^2$ with $\|f\| > 0$, the spectral measure $d\|E^\kappa f\|^2(\lambda)$ corresponding to f includes λ_0^κ in its support.*

Proof. Since $e^{-L^\kappa} f$ is everywhere nonnegative (except perhaps at the boundary), and its associated spectral measure has the same support as $d\|E^\kappa f\|^2$, we may assume that if there were a counterexample it would not vanish, except perhaps at 0.

Suppose there is some $\lambda_* > \lambda_0^\kappa$ such that $\|E^\kappa_{\lambda^*} f\| = 0$. Then for any $h \in \mathfrak{L}^2$ with $|h| \le f$,

$$
\begin{aligned}
e^{-\lambda_* t}\|f\|^2 &\ge \int_{\lambda_*}^\infty e^{-\lambda t} d\|E^\kappa f\|^2(\lambda) \\
&= \left\| e^{-\frac{t}{2}L^\kappa} f \right\| \\
&\ge \left\| e^{-\frac{t}{2}L^\kappa} h \right\| \\
&= \int_{\lambda_0^\kappa}^\infty e^{-\lambda t} d\|E^\kappa h\|^2(\lambda).
\end{aligned}
$$

Thus, it must be that $d\|E^\kappa h\|^2(\lambda)$ is supported on $[\lambda_*, \infty)$ as well. Thus, for all such h we have $\|E^\kappa_{\lambda^*} h\| = 0$.

Let $f_n = f \cdot \mathbf{1}_{[n,\infty)}$. For any $\tilde\lambda \in (\lambda_0^\kappa, \lambda_*)$, by (3) f_n is in the domain of the resolvent $R_{\tilde\lambda} = (L^\kappa - \tilde\lambda)^{-1}$. Furthermore, by (5), if we choose λ_{**} large enough so that $\|\mathbb{E}_{\lambda_{**}} f\| > 0$, then

$$
\left\|R_{\tilde\lambda} f\right\|^2 = \int (\tilde\lambda - \lambda)^{-2} d\left\|E^\kappa f\right\|^2(\lambda) \ge (\tilde\lambda - \lambda_{**})^{-2} \|\mathbb{E}_{\lambda_{**}} f\|^2 > 0.
$$

Let $g_n := R_{\tilde\lambda} f_n / \|R_{\tilde\lambda} f_n\|$. Then g_n satisfies

$$
L^\kappa g_n(x) = \tilde\lambda g_n \text{ for } x \le n.
$$

(In principle, the equality holds only in the $\mathfrak{L}^2$ sense, but it becomes true for all x since both sides are in $\mathcal{D}_\alpha$, hence in particular continuous.) By the representation

$$
R_{\tilde\lambda} f_n = \int_0^\infty e^{s\tilde\lambda} e^{-sL^\kappa} f_n ds,
$$

we see that g_n is nonnegative.

The space of solutions to the ordinary differential equation $L^\kappa g = \tilde\lambda g$ satisfying boundary condition (14) is one-dimensional, so if we renormalise to

$$
\tilde g_n := \begin{cases} (1+\alpha)^{-1} g_n(0)^{-1} g_n & \text{if } \alpha < \infty, \\ g_n'(0)^{-1} g_n & \text{if } \alpha = \infty, \end{cases}
$$

we have $\tilde g_n(x) = \tilde g_{n'}(x)$ for $x \in [0, n]$ when $n \le n'$. The limit must then be identical with the function $\varphi(\tilde\lambda, \cdot)$, and is everywhere nonnegative, contradicting the characterisation of λ_0^κ in (7).

□

2.2. **Boundary conditions, recurrence, and transience.** Defining the diffusion includes a boundary condition at 0, parametrised by $p_0 \in [0,1]$ or $\alpha = 2(1/p_0 - 1) \in [0,\infty]$:

$$(1-p_0)\phi(0) = \frac{1}{2}p_0\phi'(0). \tag{14}$$

The condition $\alpha = \infty$ corresponds to instantaneous killing at 0, while $\alpha = 0$ corresponds to reflection with no killing. Intermediate parameters correspond to "slow killing" at 0, so that the process is killed when the local time at 0 reaches an exponentially distributed random variable. The operator $L^{\kappa,\alpha}$ is associated with the closure of the quadratic form $\tilde{q}^{\kappa,\alpha}$. That is, L is the selfadjoint realization of the differential expression $-\frac{1}{2\gamma}\frac{d}{dx}\big(\gamma\frac{d}{dx}\big) + \kappa$ in $\mathfrak{L}^2$ that has boundary condition (14) at 0. The quadratic form q is a Dirichlet form, and the canonically associated Markov process is a solution for the martingale problem associated to the operator L with the appropriate killing or reflection at 0. This means there exists a family of measures $(\mathbb{P}_t)_{t\in(0,\infty)}$ on the space $C([0,\infty),\mathbb{R})$ of real valued continuous functions on $[0,\infty)$ such that for every $f \in \mathfrak{L}^2$ and every $x \in (0,\infty)$ (due to the Feller property)

$$(e^{-tL}f)(x) = \mathbb{E}_x[f(X_t), T_0 > t],$$

where (X_t) is the canonical process on $C([0,\infty),\mathbb{R})$ and T_0 is a random time defined with respect to the local time at 0. (Again, if $\alpha = \infty$ then T_0 is the time of first hitting 0; if $\alpha = 0$ then $T_0 \equiv \infty$.) In this normalisation, the scale measure has density $\gamma(x)^{-1}$ with respect to Lebesgue measure.

It is a trivial consequence of the definition of natural scale that $\int^{X_t}\gamma(x)^{-1}$ is a martingale, and so that $\mathbb{P}_x\big(X_t \text{ hits 0 eventually}\big) = 1$ for $x > 0$ if and only if the scale function is infinite at ∞; that is, for $c > 0$, $\int_c^\infty \gamma(x)^{-1}dx = \infty$. When there is killing at 0, the process is recurrent only when the scale function is infinite at both ends. In analytic terms, recurrence means that the associated generator is critical (see [19] and [40]). Recall that L^κ is called critical iff there exists a unique (up to constant multiples) positive solution ψ of $L^\kappa\psi = 0$. Otherwise L^κ is called subcritical. We know from criticality theory — for example, from Theorem 3.15 of [19] — that the generator must be critical if 0 is an isolated eigenvalue. A generalization of this fact will be used in Lemma 3.3.

The semigroup e^{-tL^κ} has a probabilistic representation: We consider the product space

$$C([0,\infty)) \times [0,\infty) = \{(\omega,\xi) \in C([0,\infty)) \times [0,\infty)\}$$

endowed with the natural product σ-field. Let $(\tilde{\mathbb{P}}_x)_{x\in(0,\infty)}$ denote the family of measures which is induced by the Dirichlet form q^0. For $x \in (0,\infty)$ we define the measures

$$\tilde{\mathbb{P}}_x \otimes e^{-\xi}\,d\xi.$$

and the stopping time

$$T_\kappa(\omega,\xi) = \inf\Big\{s \ge 0 \mid \int_0^s \kappa(\omega_s)\,ds \ge \xi\Big\}.$$

If we set

$$\tau_\partial = \min\big(T_0, T_\kappa\big)$$

then we have the Feynman-Kac representation

$$(e^{-tL^\kappa}f)(x) = \tilde{\mathbb{E}}_x\left[f(X_t), \tau_\partial > t\right] = \mathbb{E}_x\left[e^{-\int_0^t \kappa(X_s)\,ds}f(X_t), T_0 > t\right] \tag{15}$$

Since we are working with the self-adjoint version of the generator (with respect to the measure Γ), the Feynman-Kac representation holds in great generality, following the derivation in [16]. We will generally omit the tilde, since it will be clear from context which measure is meant.

Let us recall the usual Feller classification (see e.g. chapter 3 in [32]) of boundary points for diffusion generators $-\frac{1}{2}\frac{d^2}{dx^2} - b(x)\frac{d}{dx}$ in an open interval $(0, r)$.

Definition 2.3. *Let $c \in (0, r)$ be given and set $\gamma(x) = e^{\int_c^x 2b(y)\,dy}$. The point r is called* ***accessible****, if $\int_c^r \gamma(y) \int_c^y \gamma(z)^{-1}\,dz\,dy < \infty$, and otherwise* ***inaccessible****. If r is an accessible boundary point, then it is called* ***regular*** *iff $\int_c^r \int_c^y \gamma(x)^{-1}\gamma(y)\,dx\,dy < \infty$. If r is accessible and $\int_c^r \gamma(y) \left(\int_c^y \gamma(x)^{-1}\,dx\right)\,dy = \infty$, then r is called an* ***exit boundary****. If r is inaccessible then it is an* ***entrance boundary****, iff $\int_c^r \int_c^y \gamma(x)^{-1}\gamma(y)\,dx\,dy < \infty$. If r is inaccessible and $\int_c^r \gamma(y) \left(\int_c^y \gamma(x)^{-1}\,dx\right)\,dy = \infty$, then r is called* ***natural****. Of course the same classification holds for 0.*

Except where otherwise indicated, we will always assume that the boundary point ∞ is inaccessible.

It is easy to check that the boundary point r is regular if and only if $\int_c^r \gamma(x)\,dx < \infty$ and $\int_c^r \gamma(x)^{-1}\,dx < \infty$. A boundary point is thus regular in the sense of Feller if and only if it is regular in the sense of Weyl (*cf.* [24]). Let us recall the relevant definition from the Weyl theory of selfadjoint extensions of singular Sturm-Liouville operators $L^\kappa = -\frac{1}{2\gamma}\frac{d}{dx}(\gamma\frac{d}{dx}) + \kappa$ in $(0, r)$, adapted to our special situation.

Definition 2.4. *Let $z \in \mathbb{C}$. We say that boundary r is of* ***limit-point type****, if there exists $c \in (0, r)$ and $z \in \mathbb{C}$, and a solution f of $(L^\kappa - z)f = 0$ such that $\int_c^r |f(y)|^2\gamma(y)\,dy = \infty$. If there exists $c \in (0, \infty)$, such that for every solution of the equation $(L^\kappa - z)f = 0$ the integral $\int_c^r |f(y)|^2\gamma(y)\,dy$ is finite, then we say that r is of* ***limit-circle type****. An analogous notation applies to the boundary point 0.*

A fundamental result in the theory of Sturm-Liouville operators is the so called Weyl-alternative, which states that exactly one of the above situations holds and that the limit-point/limit-circle classification is independent of $z \in \mathbb{C}$ (see [24]). Moreover if we are in the limit-point case at r then for every $\mathbb{C} \setminus \mathbb{R}$ there exists exactly one solution of the equation $(L^\kappa - z)f = 0$ which satisfies $\int_c^r |f(s)|^2\,\gamma(y)dy < \infty$. Roughly limit-circle case at a boundary point r means that we have to specify boundary conditions at r in order to get a selfadjoint realization, whereas in the limit-point case at r no boundary conditions at r are necessary.

2.3. **Quasi-limiting and quasi-stationary behavior.** We say that X_t **converges from the initial distribution ν to the quasistationary distribution φ on compacta** if for any positive z, and any Borel $A \subset [0, z]$

$$\lim_{t\to\infty} \mathbb{P}_\nu\big(X_t \in A \mid X_t \le z\big) = \frac{\int_A \varphi(y)\,\gamma(y)dy}{\int_0^z \varphi(y)\,\gamma(y)dy};$$

X_t **converges from the initial distribution ν to the quasistationary distribution φ** if $\int_0^\infty \varphi(y)\,\gamma(y)dy$, and for any Borel subset $A \subset [0, \infty)$

$$\lim_{t\to\infty} \mathbb{P}_\nu\big(X_t \in A \mid \tau_\partial > t\big) = \frac{\int_A \varphi(y)\,\gamma(y)dy}{\int_0^\infty \varphi(y)\,\gamma(y)dy}.$$

Finally we say that X_t **escapes from the initial distribution** ν **to infinity** if

$$\lim_{t\to\infty} \mathbb{P}_\nu\big(X_t \le z \mid \tau_\partial > t\big) = 0$$

Remark 2.5. *In the literature there is no completely standard terminology for quasistationary distributions. The probability measure* $\frac{\varphi(y)\,\Gamma(dy)}{\int_0^\infty \varphi(y)\,\gamma(y)dy}$ *described here is sometimes also called a quasi-limiting distribution. A quasistationary distribution* $\tilde\nu$ *is often defined as a probability measure* $\tilde\nu$ *supported in* $(0,\infty)$ *satisfying*

$$\mathbb{P}_{\tilde\nu}\big(X_t \in A \mid \tau_\partial > t\big) = \tilde\nu(A),\ \forall \textit{ Borel sets } A \subset (0,\infty),\, t > 0.$$

Quasilimiting distributions are also called Yaglom limits. It is not difficult to see that quasilimiting distributions are also quasistationary distributions.

2.4. **Previous results.** Observe that we have in equation (7) that $\varphi(\lambda_0^\kappa,\cdot)$ is positive. Steinsaltz and Evans [47] showed

Theorem 2.6 (Theorem 3.3 in [47])**.** *Assume that* ∞ *is a natural boundary point and that we are in the limit-point case at* ∞*. Suppose that either*

$$\liminf_{x\to\infty} \kappa(x) > \lambda_0^\kappa \quad \textit{or} \quad \limsup_{x\to\infty} \kappa(x) < \lambda_0^\kappa.$$

Then either X_t *converges to the quasistationary distribution* $\varphi(\lambda_0^\kappa,\cdot)$*, or* X_t *escapes to infinity. In the case* $\liminf \kappa(x) > \lambda_0^\kappa$*,* X_t *converges to the quasistationary distribution* $\varphi(\lambda_0^\kappa,\cdot)$ *if and only if* $\int_0^\infty \varphi(\lambda_0^\kappa,y)\,\gamma(y)dy$ *is finite.*

A priori it would not have been clear that the conditional distribution converges, and that the mass cannot split, with part of the mass remaining on a compact interval and the remainder escaping to infinity. Having recognized that there is a a dichotomy, it is natural to then seek a simple criterion for discriminating between the cases: escape or convergence. One such is given in [47], under which X_t converges to quasistationarity, namely when $\lambda_0^\kappa < K =: \lim_{t\to\infty}\kappa(t)$ together with the growth bound

$$(GB')\quad \exists\tilde b,\tilde\kappa \ge 0\,\forall y \text{ large enough }: |b(y)| \le \tilde b y \text{ and } \kappa(y) \le \tilde\kappa y$$

or the related bound

$$(GB'')\quad \exists\bar b_1,\bar b_2,\bar\kappa,\beta \ge 0\,\forall y \text{ large enough }: \bar b_1 y^\beta \ge b(y) \ge -\bar b_1 y,\ b'(y) \ge -\bar b_2 y^2 \\ \text{and } \kappa(y) \le \bar\kappa y$$

While these conditions are satisfied in many applications they are from a theoretical point of view unsatisfactory. In particular, it seems peculiar that an upper bound on the killing rate as in (GB') should be necessary. On the contrary increasing the killing rate κ should, from a heuristic point of view, only strengthen the convergence to quasistationarity.

Remark 2.7. *We make use of Theorem 2.6 only in the case* $\lambda_0^\kappa > \lim_{x\to\infty}\kappa(x)$ *and* $\Gamma((0,\infty)) < \infty$*. In the other cases we use different techniques. In the next chapter we will show that* ∞ *is always in the limit-point case. As emphasized in* [47] *in this case the heuristic behind Theorem 2.6 is quite clear, but the translation of this idea into formal mathematics is not trivial.*

3. Analytic Results

In this chapter we derive several key analytic facts about the spectra of generators and resolvents. While some of these are standard in the theory of Sturm-Liouville operators, and well known to specialists in that field, they are less familiar to probabilists, and we explain them in some detail here. In section 3.1 we show that the technical conditions for a limit-point boundary at ∞ may be weakened. Section 3.2 derives basic results linking the spectrum and speed measure. Section 3.3 presents the standard parabolic Harnack inequality in the form that we will be using. Section 3.4 applies the analytic results to convergence on compacta. Section 3.5 explains why strong conditions on the initial conditions are unnecessary. Finally, section 3.6 generalises the results to the case of an entrance boundary at ∞.

3.1. **Classification of boundary points.** We start by establishing a connection between the Feller classification and the Weyl classification of boundary points. This has already been investigated in [55] for the case $\kappa = 0$, but in this work the author introduces the notion of weak entrance boundary and shows that one is in the limit-circle case if the boundary point is of weak entrance type. We show that there are no weak entrance boundaries at ∞ by proving that ∞ is in the limit-point case if and only if it is not a regular boundary point. The proof we give is well known in the Schrödinger case (see [7] for similar ideas in a much more general context). We assume regularity of the coefficients of the Sturm-Liouville expression, although weaker assumptions would also suffice.

Lemma 3.1. *Let the Sturm-Liouville expression $\tau f(x) = -\frac{1}{\gamma(x)}\big(\gamma(x)f'(x)\big)' + \kappa(x)f(x)$ be given. Assume that γ is strictly positive and locally Lipschitz in $(0,\infty)$ and $\kappa \in \mathfrak{L}^2_{loc}([0,\infty))$ such that $\kappa(x) \ge -C|x|^2 + D$ for some constants $C, D \ge 0$. Then we are in the limit-point case at ∞.*

Proof. We can assume, without loss of generality, that $D = 0$ and that γ is continuous up to the boundary. The first assumption is obviously harmless. If γ is not continuous up to zero we can consider the differential expression in $(1,\infty)$ instead of $(0,\infty)$. This shift does not change the Weyl-classification of τ at infinity. Similarly, we may assume that the boundary condition at 0 is Dirichlet ($\alpha = \infty$), since the classification at infinity is unaffected by the boundary condition at 0.

As usual in the theory of Sturm-Liouville operators we define the maximal operator T and the minimal operator $\widetilde{T}$ associated to the differential expression τ as

$$\mathcal{D}(T) := \big\{ f \in \mathfrak{L}^2 \mid f,\, \gamma f' \text{ absolutely continuous in } (0,\infty),\ \tau f \in \mathfrak{L}^2 \big\}$$
$$Tf := \tau f \ \text{ for } f \in \mathcal{D}(T)$$

and

$$\mathcal{D}(\widetilde{T}) := \big\{ f \in \mathcal{D}(T) \mid f \text{ has compact support in } (0,\infty) \big\}, \quad \widetilde{T} f : = \tau f \ \text{ for } f \in \mathcal{D}(\widetilde{T}),$$

respectively. Let T_D be the restriction of the maximal operator T to the domain

$$\mathcal{D} = \big\{ f \in \mathcal{D}(T) \mid f(0) = 0 \big\},$$

i.e., we put Dirichlet boundary conditions at the boundary point 0.

The deficiency indices of T_0 are $(1,1)$ if the limit-point case holds at ∞, and $(2,2)$ if limit-circle holds at ∞. In the former case, the maximal symmetric extensions of T_0 are one-dimensional; in the latter case, they are two-dimensional. If T_D defines a symmetric operator — $\langle f, T_D f\rangle \in \mathbb{R}$ for every $f \in \mathcal{D}$ — then it cannot be limit-circle, since T_D would be a

symmetric three-dimensional extension of T_0, whereas the maximal symmetric extensions are two-dimensional.

It remains to show that T_D is symmetric. Let $\varphi \in C_c^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

Further we set $\varphi_k(x) = \varphi(\frac{x}{k})$ $(k \in \mathbb{N})$. This gives, for $f \in \mathcal{D}(T_D)$ and $k \in \mathbb{N}$

$$\begin{aligned}
\langle f, T_D f\rangle &= \lim_{k\to\infty} \int_0^\infty \varphi_k(x)^2 \overline{f(x)} T_D f(x)\,\gamma(x)dx \\
&= \lim_{k\to\infty} \int_0^\infty \varphi_k(x)^2 \overline{f(x)} \big[-\frac{1}{2\gamma}(\gamma f')'(x) + \kappa(x) f(x)\big]\gamma(x)\,dx \\
&= \lim_{k\to\infty} \Big[\frac{1}{2}\int_0^\infty \big(\varphi_k(x)^2\overline{f(x)}\big)'\gamma(x)dx + \int_0^\infty \varphi_k(x)^2\kappa(x)|f(x)|^2\gamma(x)dx\Big] \\
&= \lim_{k\to\infty} \Big\{\int_0^\infty \varphi_k^2(x)\big(\frac{1}{2}|f'(x)|^2 + \kappa(x)|f(x)|^2\big)\,\gamma(x)dx \\
&\qquad + \int_0^\infty \varphi_k(x)\varphi_k'(x)\overline{f(x)} f'(x)\gamma(x)\,dx\Big\}.
\end{aligned} \tag{16}$$

Observe that in the third line the boundary term $\varphi_k^2 \bar{f}\gamma f' \,|_0^\infty$ coming from the integration by parts vanishes, since γf is continuous up to 0 (since it satisfies an ODE), $f(0) = 0$ and $\varphi_k(x)$ is identically zero for x large enough.

The first term on the right-hand side is real and we have to prove that the second term converges to 0 as $k \to \infty$. We have, by the Cauchy-Schwarz inequality and the properties of the cut-off sequence (φ_k)

$$\begin{aligned}
&\Big|\int_0^\infty \varphi_k(x)\varphi_k'(x)\overline{f(x)} f'(x)\gamma(x)dx\Big| \\
&\qquad \le \Big(\int_0^\infty \varphi_k(x)^2|f'(x)|^2\gamma(x)\,dx \int_0^\infty |\varphi_k'(x)|^2|f(x)|^2\,\gamma(x)dx\Big)^{\frac{1}{2}} \\
&\qquad \le Ck^{-1}\Big(\int_0^\infty \varphi_k(x)^2|f'(x)|^2\,\gamma(x)dx \int_k^{2k} |f(x)|^2\gamma(x)\,dx\Big)^{\frac{1}{2}}.
\end{aligned} \tag{17}$$

For the first integral on the right-hand side we integrate by parts in a similar vein to (16). The assumptions on κ as well as the elementary inequality $|ab| \le a^2/4 + b^2$ imply

$$\begin{aligned}
\frac{1}{2}\int_0^\infty \varphi_k(x)^2|f'(x)|^2\gamma(x)\,dx &= \int_0^\infty \varphi_k(x)^2\overline{f(x)}(T_D f(x))\,\gamma(x)dx - \int_0^\infty \varphi_k(x)^2\kappa(x)|f(x)|^2\,\gamma(x)dx \\
&\qquad - \int_0^\infty \varphi_k\varphi_k'(x)\overline{f(x)} f'(x)\,\gamma(x)dx \\
&\le \int_0^\infty \varphi_k(x)|f(x)||T_D f(x)|\,\gamma(x)dx + C\int_0^\infty \varphi_k^2 x^2|f(x)|^2\,\gamma(x)dx \\
&\qquad + \int_0^\infty |\varphi_k(x)\varphi_k'(x)\overline{f(x)} f'(x)|\,\gamma(x)dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^\infty \varphi_k(x)|f(x)||T_D f(x)|\,\gamma(x)dx + C\int_0^\infty \varphi_k^2 x^2 |f(x)|^2\,\gamma(x)dx \\
&\qquad + \int_0^\infty \left[\frac{1}{4}|\varphi_k(x)f'(x)|^2 + |\varphi_k'(x)\overline{f(x)}|^2\right]\gamma(x)dx \\
&\leq \|f\|\|T_D f\| + C(2k)^2\|f\| \\
&\qquad + \frac{1}{4}\int_0^\infty \varphi_k(x)^2|f'(x)|^2\,\gamma(x)dx + M^2k^{-2}\|f\|.
\end{aligned}
$$

This yields

$$
\int_0^\infty \varphi_k(x)^2|f'(x)|^2\,\gamma(x)dx \leq 4\|f\|\|T_\theta f\| + 16Ck^2\|f\|^2 + 4M^2k^{-2}\|f\|^2
$$

and therefore for large k

$$
\int_0^\infty \varphi_k(x)^2|f'(x)|^2\,dx \leq C_1 + C_2k^2 \leq C_3k^2. \tag{18}
$$

Thus the inequalities (17) and (18) impliy that (observe that $f \in L^2((0,\infty),\gamma)$)

$$
\left|\int_0^\infty \varphi_k(x)\varphi_k'(x)\overline{f(x)}f'(x)\,\gamma(x)dx\right| \leq Ck^{-1}\left(C_3k^2\int_k^{2k}|f(x)|^2\,\gamma(x)dx\right)^{\frac{1}{2}} \to 0.
$$

as $k \to \infty$. This proves the assertion, and so completes the proof. $\square$

3.2. **The spectrum of Sturm-Liouville operators.** We begin with a version of the spectral theorem for self-adjoint operators on a Hilbert space, specifically adapted to Sturm-Liouville operators. A proof of it can be found in general references on the theory of Sturm-Liouville or Schrödinger operators, such as [20, 8, 58].

Let $\tau = -\frac{1}{2\gamma}\frac{d}{dx}\big(\gamma\frac{d}{dx}\big) + \kappa$ be a Sturm-Liouville expression which is regular at 0 and in the limit-point case at infinity, and let H be the self-adjoint realization of τ in $\mathfrak{L}^2$ with boundary conditions 14 at 0. Let $\varphi(z,\cdot)$ be the unique solution of the ordinary differential equation $\tau\varphi(z,\cdot) = z\varphi(z,\cdot)$ satisfying $\varphi(z,0) = 1/(1+\alpha)$ and $\varphi'(z,0) = \alpha/(1+\alpha)$.

Given a continuous function $F \in C(\mathbb{R})$, we have a corresponding maximal multiplication operator M_F on $\mathfrak{L}^2(\mathbb{R},\mu)$ defined by

$$
\begin{aligned}
\mathcal{D}(M_F) &= \Big\{g \in \mathfrak{L}^2(\mathbb{R},\mu) \text{ s.t. } gF \in \mathfrak{L}^2(\mathbb{R},\mu)\Big\}, \\
M_F(g) &= Fg.
\end{aligned}
$$

Theorem 3.2 (Weyl's Spectral Theorem)**.** *There exists a measure σ whose support is $\Sigma(H)$, such that the map taking a compactly supported function $h \in \mathfrak{L}^2$ to the function $\hat{h} \in \mathfrak{L}^2(\Sigma(H),\sigma)$, defined by*

$$
\hat{h}(\cdot) = \int_0^\infty h(x)\varphi(\cdot,x)\,\gamma(x)dx
$$

may be uniquely extended to a unitary mapping $U : \mathfrak{L}^2((0,\infty),\Gamma) \to \mathfrak{L}^2$ with the property

$$
U\,F(H)U^{-1} = M_F.
$$

The spectrum of H is simple, and $\Sigma(F(H)) = ess\,ran_\sigma(F)$.

The spectrum $\Sigma(A)$ of a self-adjoint operator A may be divided into two components: the essential spectrum $\Sigma_{ess}(A)$, comprising the limit points and eigenvalues of infinite multiplicity; and the discrete part $\Sigma_d(A)$, comprising the isolated eigenvalues of finite multiplicity. In the Sturm-Liouville case every eigenvalue has finite multiplicity (no more than 2), so the essential spectrum consists only of limit points of the spectrum. It is well known that the essential part of the spectrum of self-adjoint operators is invariant with respect to relatively compact perturbations (see Theorem 9.15 of [53]). (We recall that an operator $V : X \to X$ on the Banach space X is called relatively compact with respect to $T : X \to X$ if $\mathcal{D}(T) \subset \mathcal{D}(V)$, and if for some $z \in \mathbb{C} \setminus \Sigma(T)$ the operator $V(T-z)^{-1}$ is compact. We refer to section 9.2 of [53] for further details.)

The core of our results is contained in the following analytic lemma, which catalogs some of the key linkages among the base of the spectrum, the scale measure, and the speed measure. These take us beyond the results of Theorem 2.6, by separating the influence of the drift from the effect of the killing term. Moreover, they show clearly why the case $\lambda_0^\kappa < K$ will turn out to be easier than the case $\lambda_0^\kappa > K$. The major results — particularly Theorems 3.15, 4.1, 4.7 and 4.9 — will in essence be just unpacking these analytic results in probabilistic terminology.

Lemma 3.3. *With the above definitions,*

(i) *If* $\lim_{x\to\infty} \kappa(x) = K$, *then* $\Sigma_{ess}(L^{0,\alpha}) + K = \Sigma_{ess}(L^{\kappa,\alpha})$.
(ii) $\lambda_0 > 0$ *and* $\int_0^\infty \gamma(x)^{-1}\,dx = \infty$ *imply* $\Gamma(\mathbb{R}_+) = \int_0^\infty \gamma(x)dx < \infty$.
(iii) $\lambda_0 > 0$ *and* $\Gamma\big([0,\infty)\big) = \infty$ *imply* $\lambda_0^{0,0} > 0$.
(iv) $\lambda_0 > 0$ *and* $\Gamma\big([0,\infty)\big) = \infty$ *imply* $\lim_{r\to\infty} \frac{1}{r}\log\Gamma([0,r)) > 0$.
(v) *If* $\lambda_0^\kappa < \liminf_{x\to\infty}\kappa(x)$ *then* λ_0^κ *is a simple isolated eigenvalue with a unique positive eigenfunction.*
(vi) *If* $\alpha > 0$ *(not pure reflection at 0) or if* Γ *is infinite, then 0 is not an isolated eigenvalue of* $L^{0,\alpha}$.
(vii) *If* $\alpha > 0$ *(not pure reflection at 0) or if* Γ *is infinite, then* $\lambda_0^\kappa > \limsup_{x\to\infty}\kappa(x)$ *implies* $\lambda_0^{0,\alpha} > 0$.

Proof. **Assertion (i)** can be derived from the fact that the essential spectra of two self-adjoint operators L_1 and L_2 coincide if for some $z \in \mathbb{C} \setminus (\Sigma(L_1) \cup \Sigma(L_2))$ the difference

$$(L_1 - z)^{-1} - (L_2 - z)^{-1}$$

is a compact operator (*cf.* Theorem 9.15 of [53]). Set $\kappa_n(t) = \mathbf{1}_{[0,n]}(t)(\kappa(t) - K)$. The resolvent equation gives for $z \in \mathbb{C} \setminus \mathbb{R}$

$$(L^{\kappa_n} - z)^{-1} - (L - z)^{-1} = (L^{\kappa_n} - z)^{-1}(L - L^{\kappa_n})(L - z)^{-1} = -(L^{\kappa_n} - z)^{-1}\kappa_n(L - z)^{-1}.$$

Observe now that the operator $\kappa_n(L-z)^{-1}$ is compact; that is, the operator acting by multiplication with κ_n is relatively compact with respect to the operator L. This can be seen by considering the explicit form of the resolvent (see chapter 3.3 in [24]; similar results can be found in [11]). We have

$$[\kappa_n(L-z)^{-1}]g(x) = \kappa_n(x)\frac{1}{W(v,u)}\bigg(v(x)\int_0^x u(y)g(y)\gamma(y)dy + u(x)\int_x^\infty v(y)g(y)\gamma(y)dy\bigg).$$

where u and v are linearly independent solutions of

$$
\begin{aligned}
(\tau - z)w = 0 \text{ satisfying}&\\
u(0) = \frac{1}{1+\alpha},\ u'(0) = \frac{\alpha}{1+\alpha},\ \text{and}&\\
\int_1^\infty |v(y)|^2\,\gamma(y)dy < \infty.&
\end{aligned}
$$

Observe that here we use the fact that we are in the limit-point case at infinity. The Wronskian $W(f,g)$ of two locally absolutely continuous functions f and g is defined by

$$W(f,g)(x) = \left[f'(x)g'(x) - f'(x)g(x)\right]\gamma(x),$$

Thus $\kappa_n(L-z)^{-1}$ is an integral operator in $\mathfrak{L}^2$ with kernel $k(\cdot,\cdot)$ given by

$$k(x,y) = \begin{cases} W(v,u)^{-1}\kappa_n(x)v(x)u(y) & \text{if } y \le x, \text{ and} \\ W(v,u)^{-1}\kappa_n(x)v(y)u(x) & \text{if } y \ge x. \end{cases}$$

The known properties of u and v imply

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty |k(x,y)|^2\,\gamma(y)\,\gamma(x)dy\,dx\\
&\quad= \frac{1}{W(v,u)^2}\int_0^n \left(|v(x)|^2\int_0^x |u(y)|^2\,\gamma(y)dy + |u(x)|^2\int_x^\infty |v(y)|^2\,\gamma(y)dy\right)|\kappa_n(x)|^2\gamma(x)dx < \infty.
\end{aligned}
$$

Thus $\kappa_n(L-z)^{-1}$ is Hilbert-Schmidt, hence also compact.

We complete the proof by observing that the resolvent equation

$$(L^{\kappa_n+K} - z)^{-1} - (L^\kappa - z)^{-1} = (L^{\kappa_n} - z)^{-1}(\kappa - \kappa_n - K)(L-z)^{-1},$$

implies

$$
\begin{aligned}
\left\|(L^{\kappa_n} - z)^{-1} - (L^\kappa - z)^{-1}\right\| &\le \|(L^{\kappa_n} - z)^{-1}\|\|\kappa - \kappa_n - K\|_\infty\|(L^\kappa - z)^{-1}\|\\
&\le \frac{1}{(\Im z)^2}\|\kappa - \kappa_n - K\|_\infty \to 0
\end{aligned}
$$

as $n \to \infty$; that is, L^{κ_n+K} converges in the norm-resolvent sense to L^κ. In the second inequality we used the fact that the operator norm of the operator that acts as multiplication by a function f is just the supremum norm of f.

Assertion (ii) is contained in [35] and also follows from Theorem 1 of the recent work [41]. (In [41] somewhat stronger conditions on the drift are imposed, but these are not actually necessary for the proof.)

Assertion (iii): Because $L^{0,\alpha}$ and $L^{0,0}$ differ only in their (one-dimensional) boundary conditions, the difference $(L^{0,0}+1)^{-1} - (L^{0,\alpha}+1)^{-1}$ has one-dimensional range, so is compact. (For more details, see [53, Satz 10.17].) The bottom of the essential spectrum of $L^{0,0}$ is then strictly positive, since it coincides with the bottom of the essential spectrum of $L^{0,\alpha}$, hence is above the bottom of the full spectrum of $L^{0,\alpha}$. If $\lambda_0^{0,0} := \inf \operatorname{spec}(L^{0,0}) = 0$ then $\lambda_0^{0,0} = 0$ is necessarily an isolated eigenvalue of the operator $L^{0,0}$. Let us assume that $\lambda_0^{0,0} = 0$. The

unique (up to positive multiples) non-trivial and non-negative eigenfunction $v_N \in \mathfrak{L}^2$ associated to $\lambda_0^{0,0} = 0$ therefore solves the boundary value problem

$$L^{0,0} v_N = \lambda_0^{0,0} v_N = 0, \; v_N(0) = \frac{1}{1+\alpha} \text{ and } \gamma \frac{dv_N}{dx}(0) = \frac{\alpha}{1+\alpha}.$$

Since this ordinary differential equation has a unique solution, and since the constant function $\mathbf{1}$ is also a solution of this equation, we conclude that $v_N = \mathbf{1}$. Thus $\mathbf{1} \in \mathfrak{L}^2$, which means that $\Gamma\big((0,\infty)\big) < \infty$, contradicting our assumption that Γ is infinite. It follows that $\lambda_0^{0,0} > 0$.

Assertion (iv) follows from the above and the work of Notarantonio [38]. His result implies that the bottom of the essential spectrum of the operator $L^{0,0}$ with Neumann boundary conditions at 0 is bounded above by $\limsup_{r\to\infty} \frac{1}{r} \log \Gamma((0,r))$. This is 0 if the volume growth is subexponential. Since we have already showed that $\lambda_0^{0,0} > 0$, the result follows.

Assertion (v): Assume first that $\lim_{x\to\infty} \kappa(x)$ exists. If $\lambda_0^\kappa < \lim_{x\to\infty} \kappa(x) = K$ then an application of the result (i) shows that $L^\kappa = L + K + (\kappa - K)$ has the same essential spectrum as $L+K$. Since L is a positive operator, the bottom of the spectrum of $L+K$, hence *a fortiori* of the essential spectrum, has to be at least K, hence bigger than λ_0^κ, which implies (v).

Let us now assume only that $\liminf_{x\to\infty} \kappa(x) > \lambda_0^\kappa$. By the decomposition principle (see section 131 in [3]) it is not difficult to see that L^κ has the same essential spectrum as the operator L_a^κ ($a > 0$), defined as the selfadjoint extension of τ^κ in $\mathfrak{L}^2((a,\infty),\Gamma)$ satisfying Dirichlet boundary conditions at a. If $a_0 > 0$ and $\varepsilon > 0$ are such that $\inf_{x \geq a_0} \kappa(x) > \lambda_0^\kappa + \varepsilon$ we conclude that

$$\begin{aligned}
\inf \Sigma_{ess}(L^\kappa) &\geq \inf \Sigma(L_{a_0}^\kappa) \\
&\geq \inf_{\substack{\varphi \in C_c^\infty(a_0,\infty) \\ \|\varphi\|_{\mathfrak{L}^2((a_0,\infty),\Gamma)}=1}} \int_{a_0}^\infty |\varphi'(x)|^2 \, \gamma(x) dx + \int_{a_0}^\infty \kappa(x) |\varphi(x)|^2 \gamma(x) dx \\
&\geq \lambda_0^\kappa + \varepsilon.
\end{aligned}$$

Assertion (vi): Let f, g be continuous functions, nonnegative and not identically zero, with compact support on $(0,\infty)$, and $\lambda_1 := \inf \Sigma(L^{0,\alpha}) \setminus \{0\}$, where $\alpha > 0$. Suppose 0 is an isolated eigenvalue, so that $\lambda_1 > 0$. Then

$$\begin{aligned}
\left\langle f, e^{-tL^0} g \right\rangle &= \int_{\Sigma(L^0)} e^{-t\lambda} d \left\langle f, Eg \right\rangle (\lambda) \\
&= \langle f, E(\{0\}) g \rangle + \int_{\lambda_1}^\infty e^{-t\lambda} d \left\langle f, Eg \right\rangle (\lambda) \\
&\to \langle f, u_0 \rangle \langle g, u_0 \rangle \text{ as } t \to \infty.
\end{aligned}$$

This is positive, since u_0 may be chosen to be strictly positive.

By Lemma 2.1 there is a constant C such that for all $x \in \operatorname{supp}(f)$,

$$\begin{aligned}\left|e^{-tL^0} g(x)\right| &\leq C\left\|\sqrt{L^0} e^{-tL^0} g\right\| \\ &\leq C\left(\int_0^\infty \lambda e^{-2\lambda t} d\|E_\lambda^0 g\|^2\right)^{1/2} \\ &\leq C\|g\| e^{-\lambda_1 t} \text{ for } t \text{ sufficiently large,}\end{aligned}$$

so that

$$\left\langle f, e^{-tL^0} g\right\rangle \leq C\|g\| e^{-\lambda_1 t} \int_0^\infty f(x)\gamma(x)dx \to 0$$

as $t \to \infty$, which is a contradiction.

Assertion (vii): Suppose first that the limit K of $\kappa(x)$ exists. Intuitively, what we are saying is that when the mass in a neighborhood of 0 shrinks at a rate faster than K (what λ_0^κ measures), it is being driven by drift: Either the mass is being swept down into a region of high killing near 0, or it is being swept up away from 0. In the latter case, the drift will still cause the mass near 0 to shrink exponentially in the absence of killing; in the former case, the killing at 0 will do the job, except in the case of pure reflection at 0.

By part (i), we see that $L^\kappa = L + K + (\kappa - K)$ and $L + K$ have the same essential spectrum. In particular we conclude that $\inf \Sigma_{ess}(L) + K = \inf \Sigma_{ess}(L + K) \geq \lambda_0^\kappa$ and therefore $\inf \Sigma_{ess}(L) \geq \lambda_0^\kappa - K > 0$, so $\lambda_0 = 0$ would imply that 0 is an isolated eigenvalue. Since this is impossible, by assertion (vi), it follows that $\lambda_0 > 0$. The extension to the case when the limit does not exist goes exactly the same way as in the proof of assertion (v) above. □

Remark 3.4. *R. Pinsky has showed* [41] *that the conclusion (ii) of Lemma 3.3 can be sharpened. Assuming that absorption is certain, it was shown that*

$$\frac{1}{8A(b)} \leq \lambda_0 \leq \frac{1}{2A(b)}, \tag{19}$$

where

$$A(b) = \left(\int_x^\infty \gamma(y)dy\right)\left(\int_0^x \gamma(y)^{-1}\, dy\right).$$

Related analytic inequalities, which are usually referred to as weighted Hardy inequalities, can be found in [37]*. Indeed the results of* [37] *imply Pinsky's bounds.*

Remark 3.5. *The fact that the bottom of the spectrum is an isolated eigenvalue is also of practical interest, because in this case the associated eigenfunction can be approximated accurately by the ground states of regular Sturm-Liouville operators on bounded intervals (see the recent survey* [54]*). Such a result has recently been rederived* [51]*, in the context of approximating the minimal quasistationary distribution of a diffusion generator with discrete spectrum via interacting particle systems of Fleming-Viot type.*

Remark 3.6. *Assume that λ_0^κ is an eigenvalue with associated eigenfunction $u_{\lambda_0^\kappa} \in \mathfrak{L}^2$, which by general theory is strictly positive and simple. Then*

$$\lim_{t\to\infty} e^{\lambda_0^\kappa t} p^\kappa(t,x,y) = c\, u_{\lambda_0^\kappa}(x) u_{\lambda_0^\kappa}(y), \tag{20}$$

where c is a normalizing constant. This was proved in [45] *for the transition function of Brownian motion on Riemannian manifolds but the proof carries over without essential changes to our case.*

We will also make use of the following result which is a special case of Theorem 3.1 in [47].

Lemma 3.7 (Theorem 3.1 in [47])**.** *Let $0 \leq f \in \mathfrak{L}^2$ with compact support $supp(f) \subset [0, \infty)$ be given and let ν_f denote the measure $f(x)\gamma(x)dx$. Let L^κ be as in Lemma 3.3 and let $p^\kappa(t, \cdot, \cdot)$ denote the integral of e^{-tL^κ}. Then for arbitrary measurable bounded sets $A, B \subset (0, \infty)$*

$$\lim_{t\to\infty} \frac{\int_0^\infty f(x) \int_B p^\kappa(t,x,y)\,\gamma(y)\,\gamma(x)\,dy\,dx}{\int_0^\infty f(x) \int_A p^\kappa(t,x,y)\,\gamma(y)\gamma(x)\,dy\,dx} = \frac{\int_B \varphi(\lambda_0^\kappa, y)\,\gamma(y)dy}{\int_A \varphi(\lambda_0^\kappa, y)\,\gamma(y)dy},$$

i.e. X_t converges from the initial distributions $\frac{\nu_f}{\int_0^\infty f(s)\,\gamma(ds)}$ on compacta to the quasistationary distribution $\varphi(\lambda_0^\kappa, \cdot)$.

The above Lemma can be proved directly, using the spectral representation for Sturm-Liouville operators. The reader will see the necessary arguments later in this work in the proof of Theorem 3.15. Our first goal is to extend this result to the case of general compactly supported initial distributions ν.

We begin by deducing some consequences of Lemma 3.7. This will lead to Proposition 3.9, which is a "strong ratio limit theorem". Before we start proving the strong ratio limit theorem we explain another analytic fact which has no direct relation to spectral theory but which will turn out to be very useful.

3.3. **Local parabolic Harnack inequality.** A crucial tool for smoothing analytic information about the transition kernel between different times and sites is the local parabolic Harnack inequality, which quite general holds for second order parabolic differential equations. One version appropriate to our current purposes may be found in [31], and states that for fixed $x_0, t \in (0, \infty)$ and $R > 0$ there is a constant C such that for every weak solution u of $(\partial_t + L^\kappa)u = 0$ which is non-negative in $Q((x_0, t_0), 4R) \subset (0, \infty) \times (0, \infty)$

$$\sup_{\Theta((x_0,t_0),R/2)} u \leq C \inf_{Q((x_0,t_0),R)} u,$$

where

$$Q((x_0, t_0), R) = \{X \in \mathbb{R}^2 \mid \max\big(|x - x_0|, \sqrt{|t - t_0|}\big) < R, t < t_0\}$$

and $\Theta((x_0, t_0), R/2) = Q((x_0, t_0 - R^2), R)$. As in Theorem 10 of [14] this inequality can be applied to the transition kernel $p^\kappa(t, x, y)$ in order to prove that for every compact $K \subset (0, \infty)$ and $T > 0$ there is a constant $c = c(K, T) > 0$ such that for $t \geq T$, $x_1, x_2, x_3, x_4 \in K$

$$c^{-1}p^\kappa(t, x_1, x_2) \leq p^\kappa(t, x_3, x_4) \leq cp^\kappa(t, x_1, x_2). \tag{21}$$

Moreover the local parabolic Harnack inequality shows that there exists a locally bounded function $\zeta : (0, \infty) \to (0, \infty)$ such that for every $t \geq 1$, $y > 0$, and $x, z > 0$ satisfying $|z - x| < \frac{1}{2} \wedge \frac{|x|}{4}$

$$p^\kappa(t, x, y) \leq \zeta(x)p^\kappa(t + 1, z, y). \tag{22}$$

3.4. Strong ratio limit theorem and convergence on compacta.

Lemma 3.8. *For any fixed $x_0 \in (0,\infty)$ the family of functions*

$$\left\{[0,\infty) \times \mathbb{R}_+ \times \mathbb{R}_+ \ni (t,x,y) \mapsto \frac{p^\kappa(t+s,x,y)}{p^\kappa(s,x_0,x_0)} \mid s \geq 1\right\}$$

is relatively compact in the space $C((0,\infty)^2,\mathbb{R})$ of real-valued continuous functions on $(0,\infty)^2$, endowed with the vague topology.

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence with $1 \leq s_n \to \infty$ and set for $t \in [0,\infty)$, $x, y \in (0,\infty)$

$$r_n(t,x,y) = \frac{p^\kappa(t+s_n,x,y)}{p^\kappa(s_n,x_0,x_0)},$$

where $a \in (0,\infty)$ is fixed. The functions $(t,x,y) \mapsto r_n(t,x,y)$ $(n \in \mathbb{N})$ are solutions to the parabolic equation

$$(2\partial_t + L_x^\kappa + L_y^\kappa) r_n(t,x,y) = 0,$$

where the operator L_x^κ and L_y^κ act as L^κ on x- and y-variable, respectively. By the local parabolic Harnack inequality (see section 3.3) we conclude that for each compact set $K \subset (0,\infty)$ there exists a constant C_K such that for all $n \in \mathbb{N}$, $t \geq 0$ and $x, y, a \in K$

$$p^\kappa(t+s_n,x,y) \leq C_K p^\kappa(t+s_n,x_0,x_0)$$

By general spectral theory it is proved in [14] that $r \mapsto p^\kappa(r,x_0,x_0)$ is non-increasing. Therefore we conclude that for $t \geq 0$ and $x, y \in K$

$$\frac{p^\kappa(t+s_n,x,y)}{p^\kappa(s_n,x_0,x_0)} \leq C_K.$$

Theorem 6.28 in [31] shows that the set $\{r_n \mid n \in \mathbb{N}\}$ is locally uniformly equicontinuous. Therefore by the theorem of Arzela-Ascoli [26, Theorem 17] there exists a subsequence $(r_{n_k})_{k\in\mathbb{N}}$ which converges locally uniformly. □

This proof is modeled on Theorem 2.2 of [4]. Since that theorem assumed the operator was critical and the coefficients were Hölder-continuous, some modification was required

The analytic core of quasilimiting behavior is the convergence of ratios of transition kernels, which we state and prove here as Proposition 3.9. This will imply convergence to the quasistationary distribution on compacta, Theorem 3.15. Convergence on the whole state space will then require a consideration of the recurrence or transience, to decide whether most of the mass stays in a compact interval or escapes to infinity.

Results comparing transition probabilities at different times and sites, in the limit as time goes to infinity, are commonly referred to as strong ratio limit theorems. Strong ratio limit theorems for certain branching processes can be found in [5]. A proof of the strong ratio property for certain Markov chains on the integers was given in [27].

Proposition 3.9. *For any $a \in (0,\infty)$*

$$\lim_{s\to\infty} \frac{p(t+s,x,y)}{p(s,a,a)} = e^{-\lambda_0^\kappa t} \frac{\varphi(\lambda_0^\kappa,x)\varphi(\lambda_0^\kappa,y)}{\varphi(\lambda_0^\kappa,a)\varphi(\lambda_0^\kappa,a)}.$$

Proof. For every sequence $(s_n)_{n\in\mathbb{N}} \subset (0,\infty)$ converging to infinity we know by Lemma 3.8 that for some subsequence $(s_{n_k})_k$ of (s_n) there exists a function ψ such that

$$\frac{p^\kappa(t+s_{n_k},x,y)}{p^\kappa(s_{n_k},a,a)} \to \psi(t,x,y),$$

where the convergence is locally uniform in $[0,\infty)\times(0,\infty)^2$. Since by Lemma 7.5 in [47] (see also [14]) for every $f\in\mathfrak{L}^2$ with compact support

$$\lim_{s\to\infty}\frac{\langle e^{-(t+s)L^\kappa}f,f\rangle}{\langle e^{-sL^\kappa}f,f\rangle}=e^{-\lambda_0^\kappa t},$$

one easily concludes that

$$\psi(t,x,y)=e^{-\lambda_0^\kappa(t)}\psi(0,x,y).$$

Lemma 3.7 shows that for every $f,g,h\in C_0^\infty(0,\infty)$

$$\begin{aligned}\frac{\int_0^\infty g(y)\varphi(\lambda_0^\kappa,y)\,\gamma(y)dy}{\int_0^\infty h(y)\varphi(\lambda_0^\kappa,y)\,\gamma(y)dy}&=\lim_{k\to\infty}\frac{\int_0^\infty f(x)\int_0^\infty g(y)p^\kappa(s_{n_k},x,y)\,\gamma(y)\gamma(x)\,dy\,dx}{\int_0^\infty f(x)\int_0^\infty h(y)p^\kappa(s_{n_k},x,y)\,\gamma(y)\gamma(x)\,dy\,dx}\\&=\lim_{k\to\infty}\frac{\int_0^\infty f(x)\int_0^\infty g(y)\frac{p^\kappa(s_{n_k},x,y)}{p^\kappa(s_{n_k},x_0,x_0)}\,\gamma(y)\gamma(x)\,dy\,dx}{\int_0^\infty f(x)\int_0^\infty h(y)\frac{p^\kappa(s_{n_k},x,y)}{p^\kappa(s_{n_k},x_0,x_0)}\,\gamma(y)\gamma(x)\,dy\,dx}\\&=\frac{\int_0^\infty f(x)\int_0^\infty g(y)\psi(0,x,y)\,\gamma(y)\gamma(x)\,dy\,dx}{\int_0^\infty f(x)\int_0^\infty h(y)\psi(0,x,y)\,\gamma(y)\gamma(x)\,dy\,dx}\,.\end{aligned}$$

This implies that for $x\in(0,\infty)$, $g,h\in C_0^\infty((0,\infty)$

$$\frac{\int_0^\infty g(y)\varphi(\lambda_0^\kappa,y)\,\gamma(y)dy}{\int_0^\infty h(y)\varphi(\lambda_0^\kappa,y)\,\gamma(y)dy}\int_0^\infty h(y)\psi(0,x,y)\,\gamma(y)dy=\int_0^\infty g(y)\psi(0,x,y)\,\gamma(y)dy,$$

and hence for every h

$$\psi(0,x,y)=\varphi(\lambda_0^\kappa,y)\frac{\int_0^\infty h(z)\psi(0,x,z)\,\gamma(z)dz}{\int_0^\infty h(z)\varphi(\lambda_0^\kappa,z)\,\gamma(z)dz}$$

Due to the symmetry of $\psi(0,\cdot,\cdot)$ we conclude that for some constant $c\geq 0$

$$\psi(0,x,y)=c\,\varphi(\lambda_0^\kappa,x)\varphi(\lambda_0^\kappa,y).$$

Because of $\psi(0,a,a)=1$ we arrive at $c^{-1}=\varphi(\lambda_0^\kappa,a)\varphi(\lambda_0^\kappa,a)$. Since this is true for every subsequence, the assertion of the theorem is proved. □

Corollary 3.10. *If ν is any compactly supported initial distribution, and f a nonnegative compactly supported measurable function with $\nu[f]>0$, then for any fixed t,*

$$|\log\mathbb{E}_\nu[f(X_{t+s})]-\log\mathbb{E}_\nu[f(X_s)]|$$

is bounded for $s\in\mathbb{R}^+$.

Proof. Since $g(s):=\int\int p^\kappa(s,x,y)\gamma(y)f(y)d\nu(x)$ is positive and continuous, it suffices to show that

$$0<\liminf_{s\to\infty}\frac{g(t+s)}{g(s)}\leq\limsup_{s\to\infty}\frac{g(t+s)}{g(s)}<\infty.$$

By (21) we may find positive c such that for s sufficiently large, and any $x_*,y_*\in K:=\mathrm{supp}(\nu)\cup\mathrm{supp}(f)$,

$$c^{-1}p^\kappa(s,x_*,y_*)\Gamma[f]\leq g(s)\leq cp^\kappa(s,x_*,y_*)\Gamma[f].$$

Thus,

$$c^{-2}\frac{p^\kappa(s+t,x_*,y_*)}{p^\kappa(s,x_*,y_*)}\leq\frac{g(s+t)}{g(s)}\leq c^2\frac{p^\kappa(s+t,x_*,y_*)}{p^\kappa(s,x_*,y_*)}.$$

By Proposition 3.9 the upper and lower bounds converge to $c^2 e^{-\lambda_0^\kappa}$ and $c^{-2} e^{-\lambda_0^\kappa}$ respectively, as $s \to \infty$. $\square$

Remark 3.11. *In terms of parabolic Martin boundary theory Proposition 3.9 says that every sequence $(s_n, x) \subset (0,\infty) \times (0,\infty)$ with $\lim_{n\to\infty} s_n = \infty$ converges in the parabolic Martin topology to the parabolic Martin boundary point corresponding to the minimal parabolic function $h_{\lambda_0^\kappa}(t,x) = e^{\lambda_0^\kappa t}\varphi(\lambda_0^\kappa, x)$. The parabolic function $h_{\lambda_0^\kappa}$ must actually be invariant, since it corresponds to a point in the parabolic Martin boundary whose time coordinate is ∞.*

Remark 3.12. *The existence of strong ratio limits for general symmetric diffusion, i.e. the existence of*

$$\lim_{t\to\infty} \frac{p(t,x,y)}{p(t,x_0,y_0)},$$

where $p(t,\cdot,\cdot)$ denotes the transition kernel of the diffusion, was investigated under special conditions by Brian Davies in [14] *and is now often referred to as Davies' conjecture. In a private communication Gady Kozma disproved this conjecture by presenting a counterexample. Proposition 3.9 shows that in one dimension the Davies conjecture is true, if one boundary point is regular. It is an open question, whether the Davies conjecture generally holds in one dimension.*

Proposition 3.13. *For all positive z, including $z = \infty$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big\{X_t \le z\big\} \quad \textit{and} \quad \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big\{X_t \le z\big\}$$

are both constant in $x > 0$. Hence also

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big\{X_t \le z \,\big|\, \tau_\partial > t\big\} \quad \textit{and} \quad \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big\{X_t \le z \,\big|\, \tau_\partial > t\big\}$$

are both constant in $x > 0$.

Proof. We prove only the first statement for lim sup, the other proof being identical. Suppose we have x, x' such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big\{X_t \le z\big\} < \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{x'}\big\{X_t \le z\big\}. \tag{23}$$

We may assume without loss of generality that $|x - x'| < \frac{1}{2} \wedge \frac{|x|}{4} \wedge \frac{|x'|}{4}$. (If not, then there must be other starting points closer together where the limits differ.) Applying (22), we have for all $t \ge 1$,

$$\mathbb{P}_{x'}\big\{X_t \le z\big\} \le \zeta(x')\mathbb{P}_x\big\{X_{t+1} \le z\big\},$$

so that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{x'}\big\{X_t \le z\big\} \le \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big\{X_t \le z\big\} + \limsup_{t\to\infty} \frac{1}{t} \log \frac{\mathbb{P}_x\big\{X_{t+1} \le z\big\}}{\mathbb{P}_x\big\{X_t \le z\big\}}$$

The limits on the second line are 0 by Corrolary 3.10. We have then a contradiction to (23), which completes the proof. $\square$

3.5. **Conditions on the initial distribution.** In their version of the ratio limit theorem [47, Theorem 3.1], Steinsaltz and Evans had to pose an additional condition on the initial distribution ν and they stated the general case as an open problem. Their most general condition reads

$$(ID') \quad \text{If } X_0 \text{ has distribution } \nu, \text{ then } \exists s \geq 0 \text{ for which the distribution of } X_s \text{ has a density } f \in \mathfrak{L}^2, \text{ with } \liminf_{\lambda \downarrow \lambda_0^\kappa} Uf(\lambda) > -\infty.$$

It is not obvious how to verify that a given initial distribution ν with compact support satisfies the condition (ID'). Using some results from spectral theory and several ideas of [47], we can remove this restriction. An essential ingredient in the proof is Lemma 2.1, which allows us to ignore the upper end of the spectrum for large t.

Lemma 3.14. *Given $g \in \mathfrak{L}^2$, we have*

$$\lim_{t\to\infty} t^{-1} \log \left\| e^{-tL^\kappa} g \right\| = -\lambda_g. \tag{24}$$

Proof. By the spectral theorem (5) we know that

$$\begin{aligned}
\limsup_{t\to\infty} t^{-1} \log \left\| e^{-tL^\kappa} g \right\|^2 &= \limsup_{t\to\infty} t^{-1} \log \int_{\lambda_g}^{\infty} e^{-2t\lambda} d \left\| E^\kappa_\lambda g \right\|^2 (\lambda) \\
&\leq -2\lambda_g + \limsup_{t\to\infty} t^{-1} \log \|g\|^2 \\
&= -2\lambda_g
\end{aligned}$$

For the lower bound we take any $\lambda_* > \lambda_g$, and have

$$\begin{aligned}
\liminf_{t\to\infty} t^{-1} \log \left\| e^{-tL^\kappa} g \right\|^2 &\geq \liminf_{t\to\infty} t^{-1} \log \int_{\lambda_g}^{\lambda_*} e^{-2t\lambda} d \left\| E^\kappa_\lambda g \right\|^2 (\lambda) \\
&\geq -2\lambda_* + \liminf_{t\to\infty} t^{-1} \log \left\| E^\kappa \big([\lambda_g, \lambda_*] \big) g \right\|^2 .
\end{aligned}$$

Since λ_g is in the support of $d\|E^\kappa g\|$, this is equal to $-2\lambda_*$. Since this is true for any $\lambda_* > \lambda_g$, this completes the proof of (24). ☐

For a Radon measure ν on $(0, \infty)$ and a Borel measurable function $f : (0, \infty) \mapsto \mathbb{C}$ we use the notation $\langle \nu, f \rangle := \int_0^\infty f(s)\, \nu(ds)$.

Theorem 3.15. *The killed diffusion X_t converges on compacta to the quasistationary distribution with density proportional to $\varphi(\lambda_0^\kappa, \cdot)$ from any initial distribution which is compactly supported in $(0, \infty)$.*

Proof. An application of Weyl's eigenfunction expansion theorem and Fubini's theorem tells us that the operator $E^\kappa([\lambda_0^\kappa, \lambda_1))e^{-tL^\kappa}$ has a continuous integral kernel

$$(t, x, y) \mapsto h^{\lambda_1}(t, x, y) = \int_{[\lambda_0^\kappa, \lambda_1]} e^{-t\lambda} \varphi(\lambda, x) \varphi(\lambda, y)\, d\sigma(\lambda) \tag{25}$$

with respect to the measure Γ. This implies that for every compact subset $K \subset [0, \infty)$ the function $E^\kappa([\lambda_0^\kappa, \lambda_1])e^{-tL^\kappa} \mathbf{1}_K$ is continuous and therefore

$$\langle \nu, E^\kappa([\lambda_0^\kappa, \lambda_1])e^{-tL^\kappa} \mathbf{1}_K \rangle = \int_{\mathbb{R}} E^\kappa([\lambda_0^\kappa, \lambda_1])e^{-tL^\kappa} \mathbf{1}_K(x)\, d\nu(x)$$

is well defined. For every Borel set $A \subset [0,z]$, then,

$$\nu\Big[E^\kappa([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\Big] = \int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\left[\int_0^z \varphi(\lambda,x)\,d\nu(x)\int_A \varphi(\lambda,y)\,d\Gamma(y)\right]\,d\sigma(\lambda). \tag{26}$$

Let $g:[\lambda_0^\kappa,\infty)\to\mathbb{R}$ be any continuous function. Then

$$\limsup_{t\to\infty}\left|\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}g(\lambda)\,d\sigma(\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\,d\sigma(\lambda)} - g(\lambda_0^\kappa)\right| \le \sup_{[\lambda_0^\kappa,\lambda_1]}|g(\lambda)-g(\lambda_0^\kappa)|. \tag{27}$$

As in the proof of Theorem 3.1 of [47], for any $\lambda_1,\tilde\lambda_1,\lambda_2>\lambda_0^\kappa$ set $\lambda_*=\lambda_1\wedge\tilde\lambda_1\wedge\lambda_2\wedge\tilde\lambda_2$ and $\lambda^*=\lambda_1\vee\tilde\lambda_1\vee\lambda_2\vee\tilde\lambda_2$. Then we have the bound

$$\left|\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}g(\lambda)\,d\sigma(\lambda)}{\int_{[\lambda_0^\kappa,\lambda_2]} e^{-t\lambda}\,d\sigma(\lambda)} - \frac{\int_{[\lambda_0^\kappa,\tilde\lambda_1]} e^{-t\lambda}g(\lambda)\,d\sigma(\lambda)}{\int_{[\lambda_0^\kappa,\tilde\lambda_2]} e^{-t\lambda}\,d\sigma(\lambda)}\right| \le e^{(\lambda_0^\kappa-\lambda_*)t}\frac{\int_{[\lambda_0^\kappa,\lambda^*]}|g(\lambda)|d\sigma(\lambda)}{\int_{[\lambda_0^\kappa,\lambda_*)]}d\sigma(\lambda)},$$

which tells us that

$$\limsup_{t\to\infty}\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}g(\lambda)\,d\sigma(\lambda)}{\int_{[\lambda_0^\kappa,\lambda_2]} e^{-t\lambda}\,d\sigma(\lambda)} \qquad \text{is independent of } \lambda_1,\lambda_2\in(\lambda_0^\kappa,\infty). \tag{28}$$

Since g is continuous, (27) and (28) combine to show that

$$\lim_{t\to\infty}\left|\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}g(\lambda)\,d\sigma(\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\,d\sigma(\lambda)} - g(\lambda_0^\kappa)\right| = 0. \tag{29}$$

By (26) we now see that for every $\lambda_1\in(\lambda_0^\kappa,\infty)$

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\nu[E^\kappa([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A]}{\nu[E^\kappa([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_{[0,z]}]}\\
&\qquad\qquad = \lim_{t\to\infty}\frac{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\left[\int_0^z \varphi(\lambda,x)\,d\nu(x)\int_A\varphi(\lambda,y)\,d\Gamma(y)\right]\,d\sigma(\lambda)}{\int_{[\lambda_0^\kappa,\lambda_1]} e^{-t\lambda}\left[\int_0^z \varphi(\lambda,x)\,d\nu(x)\int_0^z\varphi(\lambda,y)\,d\Gamma(y)\right]\,d\sigma(\lambda)}\\
&\qquad\qquad = \frac{\int_A\varphi(\lambda_0^\kappa,y)\,\gamma(y)dy}{\int_0^z\varphi(\lambda_0^\kappa,y)\,\gamma(y)dy}.
\end{aligned} \tag{30}$$

The assertion of the theorem follows immediately from (30) once it is shown that

$$\lim_{t\to\infty}\frac{\mathbb{P}_\nu(X_t\in A)}{\mathbb{P}_\nu(X_t\le z)} = \lim_{t\to\infty}\frac{\nu\left[E^\kappa([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\right]}{\nu\left[E^\kappa([\lambda_0^\kappa,\lambda_1])e^{-tL^\kappa}\mathbf{1}_{[0,z]}\right]}. \tag{31}$$

Observe that

$$\begin{aligned}
\frac{\nu\left[e^{-tL^\kappa}\mathbf{1}_A\right]}{\nu\left[E^\kappa([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\right]} &= \frac{\nu\left[E^\kappa([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\right]+\nu\left[E^\kappa((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A\right]}{\nu\left[E^\kappa([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\right]}\\
&= 1+\frac{\nu\left[E^\kappa((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A\right]}{\nu\left[E^\kappa([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\right]}.
\end{aligned} \tag{32}$$

Since $e^{-tL^\kappa}\mathbf{1}_A$ and $E^\kappa([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A$ are continuous, the function $E^\kappa((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A$ must also be continuous. Thus $\nu[E^\kappa((\lambda_1,\infty))e^{-tL^\kappa}\mathbf{1}_A]$ is well defined. By Lemma 2.1,

$$-\lim_{t\to\infty}\frac{1}{t}\log\left|\langle\nu,E^\kappa\big((\lambda_1,\infty)\big)e^{-tL^\kappa}\mathbf{1}_A\rangle\right| \ge \lambda_1. \tag{33}$$

As $\varphi(\lambda_0^\kappa, x) > 0$ for every $x \in (0,\infty)$ there is, by continuity, $\lambda_1 > \lambda_0^\kappa$ such that for every $\lambda \in [\lambda_0^\kappa, \lambda_1]$

$$\int_0^\infty \varphi(\lambda,x)d\nu(x) \text{ and } \int_A \varphi(\lambda,y)\gamma(y)dy \text{ are both positive.}$$

Then it is easy to see that

$$-\lim_{t\to\infty}\frac{1}{t}\log\int_{[\lambda_0^\kappa,\lambda_1]} e^{-\lambda t}\int_0^\infty \varphi(\lambda,x)d\nu(x)\int_A \varphi(\lambda,y)\,\gamma(y)dyd\sigma(\lambda) \le \lambda_0^\kappa. \tag{34}$$

Equation (32), (33) and (34) combine to show

$$\lim_{t\to\infty}\frac{\nu\left[e^{-tL^\kappa}\mathbf{1}_A\right]}{\nu\left[E^\kappa([0,\lambda_1])e^{-tL^\kappa}\mathbf{1}_A\right]} = 1,$$

and therefore (31). □

3.6. **Entrance boundary at ∞.** As mentioned above, with exception of the recent [9], work on these problems has generally assumed that 0 is regular and ∞ natural. Intuitively, we should expect the problems to be easier if ∞ is an entrance boundary. We show that this is indeed the case in Theorem 3.16, as the spectrum of the operator L^κ is purely discrete. This interesting fact has not been mentioned by previous authors (*cf.* section 3 in [9]) working on quasistationary distributions for one-dimensional diffusions. The proof relies on standard ideas from the spectral theory of differential operators.

Theorem 3.16. *If ∞ is an entrance boundary, then the spectrum of L^κ is discrete.*

Proof. Assume that 0 is a regular boundary point, and we begin by considering the case $\kappa \equiv 0$. Let f be a solution to the eigenvalue equation $-\frac{1}{2\gamma}(\gamma f')' = \lambda f$ on $(0,\infty)$, for some $\lambda > 0$.

Let $x > 1$ be any local maximum, and $\tilde{x} > x$ the first local minimum following x (assuming there is one). Since f solves the equation $\tau u = \lambda u$ one easily sees that local maxima are positive and local minima negative. Integrating by parts and using the fact that $f'(x) = f'(\tilde{x}) = 0$, we have

$$\begin{aligned}
0 < f(x) - f(\tilde{x}) &= \int_{\tilde{x}}^x f'(y)dy \\
&= \int_{\tilde{x}}^x (\gamma(y)f'(y))\gamma(y)^{-1}dy \\
&= \int_x^{\tilde{x}} (\gamma(y)f'(y))'\int_1^y \gamma(z)^{-1}dzdy \\
&= -2\lambda\int_x^{\tilde{x}} \gamma(y)f(y)\int_1^y \gamma(z)^{-1}dzdy \\
&< 2\lambda\big(f(x)-f(\tilde{x})\big)\int_x^{\tilde{x}}\gamma(y)\int_1^y \gamma(z)^{-1}dzdy,
\end{aligned}$$

from which we conclude that

$$\frac{1}{2\lambda} < \int_x^{\tilde{x}}\gamma(y)\int_1^y \gamma(z)^{-1}dzdy.$$

Since we have assumed that ∞ is an entrance boundary, we know that

$$\begin{aligned}\infty &> \int_1^\infty \gamma(y) \int_1^y \gamma(z)^{-1} dz dy \\ &\geq \sum_{\text{pairs } (x,\tilde{x})} \int_x^{\tilde{x}} \gamma(y) \int_1^y \gamma(z)^{-1} dz dy \\ &\geq (2\lambda)^{-1} \cdot \#\text{pairs } (x, \tilde{x}).\end{aligned}$$

Since the zeroes of f are separated by alternating local minima and maxima, it follows that f has only finitely many zeroes on $(1, \infty)$, hence only finitely many zeroes in all. It follows from a theorem of P. Hartmann [52, Theorem 1.1] that the spectrum of L^0 (the operator with $\kappa \equiv 0$) is discrete.

Suppose now that the spectrum of L^κ is not discrete. Then there is a λ_* such that E_{λ_*} has infinite-dimensional range. All such x are in the domain of q^κ and satisfy $q^\kappa(x, x) \leq \lambda_* \|x\|^2$. But then they are also in the domain of q^0 and satisfy $q^0(x, x) \leq \lambda_* \|x\|^2$. By the minimax principle for the discrete spectrum (*cf.* [53, Theorem 8.8]), this contradicts the fact that L has discrete spectrum. □

Remark 3.17. *There are general necessary and sufficient conditions for the discreteness of the spectrum of Sturm-Liouville operators obtained in* [13]*, of which Theorem 3.16 may be seen as a special case. However, general versions found in the literature, such as the main result in* [13] *and Theorem 1 in* [41]*, do not seem to be immediately applicable.*

4. Convergence to Quasistationarity

In this section we consider the problem of convergence to the Yaglom limit. More precisely we ask for conditions, which ensure that X_t converges to the quasistationary distribution given by the density $\varphi(\lambda_0^\kappa, \cdot)$. Recall that we always assume that 0 is regular. Except in the final subsection 4.5 we will also assume that ∞ is a natural boundary point.

4.1. High killing at ∞.

In this section we consider the case where the asymptotic killing rate is strictly bigger than λ_0^κ. Theorem 2.6 shows that one has convergence to the quasistationary distribution if and only if the lowest eigenfunction is integrable. We give a new proof of this assertion and moreover prove that the lowest eigenfunction is actually always integrable. Therefore $\liminf \kappa > \lambda_0^\kappa$ always implies convergence to the quasistationary distribution. In contrast to [47], we do not need to assume that ∞ is a natural boundary: Since the bottom of the spectrum is an isolated eigenvalue, the corresponding eigenfunction is square-integrable, as well as λ_0^κ-invariant.

Theorem 4.1. *Suppose that* $\liminf_{x\to\infty} \kappa(x) > \lambda_0^\kappa$. *Then we have*

$$\lim_{t\to\infty} e^{\lambda_0^\kappa t} \mathbb{P}^x(\tau_\partial > t) = u_{\lambda_0^\kappa}(x) \int_0^\infty u_{\lambda_0^\kappa}(y)\, \gamma(y) dy, \tag{35}$$

where $u_{\lambda_0^\kappa} \in \mathfrak{L}^2$ *denotes the uniquely determined (up to positive multiples) eigenfunction associated to the eigenvalue* λ_0^κ. *Moreover, the process* (X_t) *associated to the Dirichlet form* q^κ *converges to the quasistationary distribution* $u_{\lambda_0^\kappa}$.

The theorem will be the direct consequence of two lemmas: Lemma 4.2, which states that quasilimiting convergence follows whenever the eigenfunction $u_{\lambda_0^\kappa}$ is in $\mathfrak{L}^2$ and $\mathfrak{L}^1$; and Lemma 4.3, which states that $u_{\lambda_0^\kappa}$ is indeed in $\mathfrak{L}^1$ when $\liminf_{x\to\infty} \kappa(x) > \lambda_0^\kappa$.

Lemma 4.2. *Suppose* $u_{\lambda_0^\kappa} \in \mathfrak{L}^1 \cap \mathfrak{L}^2$*. Then* (35) *holds.*

Proof. We know from Lemma 3.3 (part v) that λ_0^κ is an isolated eigenvalue. Therefore, the eigenfunction $u_{\lambda_0^\kappa}$ is square integrable and satisfies

$$e^{-tL^\kappa} u_{\lambda_0^\kappa} = e^{-t\lambda_0^\kappa} u_{\lambda_0^\kappa} \text{ in } \mathfrak{L}^2, \text{ hence identically (since } u_{\lambda_0^\kappa} \text{ is continuous).} \tag{36}$$

By (22), for $r > 0$ sufficiently small,

$$\begin{aligned} p^\kappa(t,x,y) &= \frac{\int_{B_r(x)} p^\kappa(t,x,y) u_{\lambda_0^\kappa}(\tilde{x})\,\gamma(\tilde{x}) d\tilde{x}}{\int_{B_r(x)} u_{\lambda_0^\kappa}(\tilde{x})\,\gamma(\tilde{x}) d\tilde{x}} \\ &\le \zeta(x) \frac{\int_{B_r(x)} p^\kappa(t+1,\tilde{x},y) u_{\lambda_0^\kappa}(\tilde{x})\,\gamma(\tilde{x}) d\tilde{x}}{\int_{B_r(x)} u_{\lambda_0^\kappa}(\tilde{x})\,\gamma(\tilde{x}) d\tilde{x}} \\ &\le \zeta(x) \frac{e^{-(t+1)\lambda_0^\kappa} u_{\lambda_0^\kappa}(y)}{\int_{B_r(x)} u_{\lambda_0^\kappa}(\tilde{x})\,\gamma(\tilde{x}) d\tilde{x}} \end{aligned}$$

For fixed x, $p^\kappa(t,x,y)e^{t\lambda_0^\kappa}$ is dominated by a constant times $u_{\lambda_0^\kappa}(y)$, which is in $\mathfrak{L}^1$. The dominated convergence theorem, together with (20), implies that there is a constant c such that for any Borel set U,

$$\begin{aligned} \lim_{t\to\infty} e^{\lambda_0^\kappa t} \mathbb{P}_x\big(X_t \in U, \tau_\partial > t\big) &= \lim_{t\to\infty} \int_0^\infty e^{\lambda_0^\kappa t} p^\kappa(t,x,y) \mathbf{1}_U(y)\,\gamma(y) dy \\ &= c u_{\lambda_0^\kappa}(x) \int_U u_{\lambda_0^\kappa}(y)\,\gamma(y) dy. \end{aligned} \tag{37}$$

Taking quotients,

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}^x\big(X_t \in U \mid \tau_\partial > t\big) &= \lim_{t\to\infty} \frac{\mathbb{P}^x\big(X_t \in U, \tau_\partial > t\big)}{\mathbb{P}^x\big(\tau_\partial > t\big)} \\ &= \frac{\lim_{t\to\infty} e^{\lambda_0^\kappa t}\mathbb{P}^x\big(X_t \in U, \tau_\partial > t\big)}{\lim_{t\to\infty} e^{\lambda_0^\kappa t}\mathbb{P}^x\big(\tau_\kappa > t\big)} \\ &= \frac{c\int_U u_{\lambda_0^\kappa}(y)\,\gamma(y) dy}{c\int_0^\infty u_{\lambda_0^\kappa}(y)\,\gamma(y) dy}. \end{aligned}$$

□

For the second part of the proof we apply an argument used in [8] to derive properties of eigenfunctions of Schrödinger operators. Some modification is required to deal with the complication that we have a domain with boundary, and we do not know *a priori* that the eigenfunctions are bounded. The one-dimensional setting helps us to overcome these complications.

Lemma 4.3. *Assume that* $\lambda_0^\kappa < K := \liminf_{x\to\infty} \kappa(x)$*. Then the square integrable non-negative eigenfunction* $u_{\lambda_0^\kappa}$ *associated to the isolated eigenvalue* λ_0^κ *is integrable with respect to the measure* Γ*.*

Proof. By (36) and the Feynman-Kac formula

$$e^{-\lambda_0^\kappa t} u_{\lambda_0}(x) = \mathbb{E}_x\left[e^{-\int_0^t \kappa(X_s^*)\,ds} u_{\lambda_0^\kappa}(X_t^*), T_0 > t\right], \tag{38}$$

for every $x \in [0,\infty)$, where X_s^* is the diffusion which is killed only at the boundary. For $t \geq 0$ we define the martingale

$$M_t = e^{-\int_0^t (\kappa-\lambda_0^\kappa)(X_s^*)\,ds} u_{\lambda_0^\kappa}(X_t^*)\mathbf{1}_{\{T_0>t\}}.$$

By the assumption $\lambda_0^\kappa < K$ there exist positive real numbers a and ε such that $\kappa(x) - \lambda_0^\kappa > \varepsilon$ for every $x \in [a,\infty)$. Let T_a be the first hitting time of the set $[0,a]$.

By the optional sampling theorem we get for every $T > 0$ and $x > a$

$$\begin{aligned}
u_{\lambda_0^\kappa}(x) &= \mathbb{E}_x\left[e^{-\int_0^{T_a\wedge T}(\kappa-\lambda_0^\kappa)(X_s^*)\,ds} u_{\lambda_0^\kappa}(X^*_{T_a\wedge T})\mathbf{1}_{\{T_0>T_a\wedge T\}}\right]\\
&= \mathbb{E}_x\left[e^{-\int_0^{T}(\kappa-\lambda_0^\kappa)(X_s^*)\,ds} u_{\lambda_0^\kappa}(X^*_{T})\mathbf{1}_{\{T_a>T\}}\right]\\
&\qquad + \mathbb{E}_x\left[e^{-\int_0^{T_a}(\kappa-\lambda_0^\kappa)(X_s^*)\,ds} u_{\lambda_0^\kappa}(a)\mathbf{1}_{\{T_a\leq T\}}\right]\\
&\leq e^{-\varepsilon T}\mathbb{E}_x\left[u_{\lambda_0^\kappa}(X_T^*)\mathbf{1}_{\{T_0>T\}}\right] + u_{\lambda_0^\kappa}(a)\mathbb{E}_x\left[e^{-\varepsilon T_a\wedge T}\right].
\end{aligned}\tag{39}$$

By Lemma 2.1 and the Spectral Theorem (5) the first term is bounded by

$$\begin{aligned}
&e^{-\varepsilon T}C_\alpha(x)\|\sqrt{L}e^{-TL}u_{\lambda_0^\kappa}\| + C'_\alpha(x)\|e^{-TL}u_{\lambda_0^\kappa}\|\\
&\qquad = e^{-\varepsilon T}\left[C_\alpha(x)\left(\int_0^\infty \lambda e^{-2T\lambda}d\|E^0 u_{\lambda_0^\kappa}\|^2(\lambda)\right)^{1/2}\right.\\
&\qquad\qquad\left. + C'_\alpha\left(\int_0^\infty e^{-2T\lambda}d\|E^0 u_{\lambda_0^\kappa}\|^2(\lambda)\right)^{1/2}\right]\\
&\qquad \leq e^{-\varepsilon T}2T^{-1/2}\|u_{\lambda_0^\kappa}\|\\
&\qquad \xrightarrow{T\to\infty} 0.
\end{aligned}\tag{40}$$

We have then, from (39) and the Dominated Convergence Theorem, that

$$\begin{aligned}
0 \leq u_{\lambda_0^\kappa}(x) &\leq \lim_{T\to\infty} u_{\lambda_0^\kappa}(a)\mathbb{E}_x\left[e^{-\varepsilon T_a\wedge T}\right]\\
&= u_{\lambda_0^\kappa}(a)\mathbb{E}_x\left[e^{-\varepsilon T_a}\right]
\end{aligned}\tag{41}$$

We now appeal to a basic fact from potential theory (stated and proved in much greater generality as Proposition D.15 of [16]; see also p. 285 of [6]): There is a constant $C(a)$ such that for all $x \geq a$,

$$\mathbb{E}_x\left[e^{-\varepsilon T_a}; T_a < \infty\right] = C(a)g^\varepsilon(x,a),\tag{42}$$

where g^ε is the ε-potential, defined by

$$g^\varepsilon(x,y) = \int_0^\infty e^{-\varepsilon t}p(t,x,y)\,dt,$$

where $p(t,x,y)=p^0(t,x,y)$ denotes the integral kernel of the operator e^{-tL}. Since e^{-tL} is self-adjoint, the integral kernel $p(t,x,y)$ is symmetric with respect to Γ, so that

$$\begin{aligned}\int_a^\infty u_{\lambda_0^\kappa}(x)\gamma(x)dx &\le u_{\lambda_0^\kappa}(a)\int_a^\infty \mathbb{E}_x\big[e^{-\varepsilon T_a}\big]\gamma(x)dx\\ &\le C(a)u_{\lambda_0^\kappa}(a)\int_0^\infty g^\varepsilon(x,a)\gamma(x)dx\\ &= C(a)u_{\lambda_0^\kappa}(a)\int_0^\infty\int_0^\infty e^{-\varepsilon t}p(t,x,a)\,\gamma(x)dx\,dt\\ &= C(a)u_{\lambda_0^\kappa}(a)\int_0^\infty\int_0^\infty e^{-\varepsilon t}p(t,a,x)\,\gamma(x)dx\,dt\\ &\le C(a)\varepsilon^{-1}u_{\lambda_0^\kappa}(a).\end{aligned}$$

Since $u_{\lambda_0^\kappa}(x)\gamma(x)$ is bounded on $[0,a]$, this completes the proof. □

Remark 4.4. *The above result reflects a general principle, which seems to be well-known to analysts and mathematical physicists: The decay of the eigenfunctions associated with isolated eigenvalues is dictated by the decay of the Green's function, at least in regions where the potential κ is negligible .*

4.2. **Low killing at ∞: The recurrent case.** We assume for the remainder of this section that $K:=\lim_{x\to\infty}\kappa(x)$ exists. Whereas the total surviving mass in the case $K>\lambda_0^\kappa$ declines at the strictly exponential rate $e^{-\lambda_0^\kappa t}$, in the case $\lim_{x\to\infty}\kappa(x)<\lambda_0^\kappa$ one typically has

$$\lim_{t\to\infty} e^{\lambda_0^\kappa t}\,\mathbb{P}_x(X_t\in A,\tau_\partial>t)=0 \tag{43}$$

for every bounded Borel set $A\subset[0,\infty)$. (This can be seen for a Brownian motion with constant drift by direct computation.) Equation (43) remains true for every diffusion, if the bottom of the spectrum of the diffusion generator is not an eigenvalue in the $\mathfrak{L}^2$-sense. Thus we cannot rely upon arguments that assume a spectral gap.

It may seem surprising that, despite the complicated relationship between the unkilled motion and killing for determining the lifetime of the process (and hence, whether it returns to its starting point), the conventional transience/recurrence dichotomy for the *unkilled* process is exactly the criterion that distinguishes between convergence and escape to infinity. We begin in this section by assuming that the unkilled process is recurrent, which is equivalent to assuming that $\int_0^\infty\gamma(x)^{-1}\,dx=\infty$, and show that this implies convergence to quasistationarity. In particular the lowest eigenfunction $\varphi(\lambda_0^\kappa,\cdot)$ is integrable (but now not square integrable) with respect to Γ. In section 4.3 we then address the case when the unkilled process is transient.

As in [47] we define, for Borel sets A, the family of measures

$$F_t(\nu,A)=\mathbb{P}_\nu\big(X_t\in A\mid\tau_\partial>t\big)$$

and

$$a_t(\nu,r)=\mathbb{P}_\nu\big(\tau_\partial>t+r\mid\tau_\partial>t\big)=\int F_t(\nu,dy)\mathbb{P}_y\big(\tau_\partial>r\big).$$

If the process X_t started from the compactly supported initial distribution ν escapes to infinity, then for any sequence $(t_n)_{n\in\mathbb{N}}$ converging to infinity the measures $F_{t_n}(\nu,\cdot)$ converge weakly to point the measure δ_∞. If the process X_t started from ν converges to the quasistationary disdistribution φ then then the limit of $F_{t_n}(\nu,dy)$ is concentrated on $\mathbb{R}_+$, and has

the density $\frac{\varphi(\lambda_0^\kappa,\cdot)}{\int_0^\infty \varphi(\lambda_0^\kappa,y)\,\gamma(y)dy}$ with respect to Γ. The next lemma is in essence a combination of Lemma 5.3 and Theorem 3.3 in [47], together with our Lemma 3.7.

Lemma 4.5. *Assume that ∞ is a natural boundary point and suppose that $\lambda_0^\kappa \neq K$. Then the limit $a(\nu,r) = \lim_{t\to\infty} a_t(\nu,r)$ exists, and satisfies*

$$a(\nu,r) = F(\nu,\mathbb{R}_+)\int \varphi(\lambda_0^\kappa,y)\mathbb{P}_y\big(\tau_\partial > r\big)\,\gamma(y)dy + \big(1-F(\nu,\mathbb{R}_+)\big)e^{-Kr}. \tag{44}$$

Either $F(\nu,\mathbb{R}_+)=0$ for every compactly supported initial distribution ν or $F(\nu,\mathbb{R}_+)=1$ for every such ν.

*There exists $\eta_\nu\in\mathbb{R}$ (called the **asymptotic mortality rate**) such that*

$$a(\nu,r) = e^{-\eta_\nu r}. \tag{45}$$

If the process escapes to infinity then $\eta_\nu = K$.

Proof. Let ν be a compactly supported initial distribution. Let $(t_n)\subset(0,\infty)$ be a sequence converging to infinity. On the compactification $[0,\infty]$ of $(0,\infty)$ the sequence of measures $F_{t_n}(\nu,dy)$ has a limit point. By Theorem 2.6 this limit point is either a measure on $(0,\infty)$ which has the density $\frac{\varphi(\lambda_0^\kappa,\cdot)}{\int_0^\infty \varphi(\lambda_0^\kappa,y)\,\gamma(y)dy}$ with respect to the measure Γ or is the point mass at ∞. Theorem 2.6 shows that there is only one limit point and that the limit point is independent of the sequence (t_n) and the initial distribution ν. Thus $F_t(\nu,dy)$ converges weakly. If ∞ is natural, then

$$\lim_{y\to\infty}\mathbb{P}_y\big(\tau_\partial > r\big) = e^{-Kr}.$$

This shows that

$$\lim_{t\to\infty}\int F_t(\nu,dy)\mathbb{P}_y\big(\tau_\partial>r\big) = F(\nu,\mathbb{R}_+)\int \varphi(\lambda_0^\kappa,y)\mathbb{P}_y\big(\tau_\partial > r\big)\,\gamma(y)dy + \big(1-F(\nu,\mathbb{R}_+)e^{-Kr}\big),$$

which is (44).

For any $r,s\geq 0$ we have

$$\begin{aligned}
a(\nu,r+s) &= \lim_{t\to\infty}\frac{\mathbb{P}_\nu(\tau_\partial > t+r+s)}{\mathbb{P}(\tau_\partial > t_n)}\\
&= \lim_{t\to\infty}\frac{\mathbb{P}_\nu(\tau_\partial > t+r+s)}{\mathbb{P}_\nu(\tau_\partial > t+s)}\frac{\mathbb{P}_\nu(\tau_\partial > t+s)}{\mathbb{P}_\nu(\tau_\partial > t)}\\
&= \lim_{n\to\infty}\frac{\mathbb{P}_\nu(\tau_\partial > n+r+s)}{\mathbb{P}_\nu(\tau_\partial > n+s)}\lim_{n\to\infty}\frac{\mathbb{P}_\nu(\tau_\partial > n+s)}{\mathbb{P}_\nu(\tau_\partial > n)}\\
&= a(\nu,r)a(\nu,s),
\end{aligned}$$

which directly implies (45). The final statement follows directly from (44). $\square$

In order to decide, whether X_t converges to the quasistationary distribution, we investigate the asymptotic behavior of the function $r\mapsto\mathbb{P}_\nu\big(\tau_\partial > r\big)$, as $r\to\infty$.

Lemma 4.6. *Suppose that the lowest eigenvalue λ_0^κ is strictly positive and that Γ is a finite measure. Then for any compactly supported initial distribution ν we have*

$$-\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}_\nu\big(\tau_\partial > t\big) = \lambda_0^\kappa$$

Proof. Since Γ is finite, the constant function $\mathbf{1}$ is in $\mathfrak{L}^2$. Lemma 2.1 implies

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}_\nu\big(\tau_\partial>t\big)=\limsup_{t\to\infty}\frac{1}{t}\log\int e^{-tL^\kappa}\mathbf{1}(x)d\nu(x)\\ \leq-\lambda_0^\kappa. \tag{46}$$

Now we need a corresponding lower bound. Fix any $z>0$, and let $\mathcal{I}:=\{x:|x-z|<\frac{1}{2}\wedge\frac{z}{4}\}$. By (22) there is a constant $C(z)=\sup_{w\in\mathcal{I}}\zeta(w)$ such that

$$\begin{aligned}\big\|e^{-\frac{t}{2}L^\kappa}\mathbf{1}_\mathcal{I}\big\|^2&=\big\langle e^{-tL^\kappa}\mathbf{1}_\mathcal{I},\mathbf{1}_\mathcal{I}\big\rangle\\&=\int_\mathcal{I}\int_\mathcal{I}p^\kappa(t,x,y)\,d\Gamma(y)\,d\Gamma(x)\\&\leq C(z)\int_\mathcal{I}p^\kappa(t+1,z,y)\,\gamma(y)dy\\&\leq C(z)\,\mathbb{P}_z\big(\tau_\partial>t+1\big).\end{aligned}\tag{47}$$

By Lemma 3.14 and Lemma 2.2 we see that

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}_z\big(\tau_\partial>t\big)\geq-\inf\operatorname{supp}d\|E^\kappa\mathbf{1}_\mathcal{I}\|^2(\lambda)=-\lambda_0^\kappa,$$

Since the exponential rate of decay of $\mathbb{P}_z\big(\tau_\partial>t\big)$ is locally constant in z, by Proposition 3.13, this completes the proof. □

Theorem 4.7. *Suppose that $K<\lambda_0^\kappa$, and $\int_0^\infty\gamma(x)^{-1}\,dx=\infty$. Then X_t started from an arbitrary compactly supported initial distribution ν converges to the quasistationary distribution with Γ-density proportional to $\varphi(\lambda_0^\kappa,\cdot)$. Moreover, the asymptotic mortality rate η_ν is independent of ν and equals λ_0^κ.*

Proof. If X_t escapes to infinity then we know from Lemma 4.5 that

$$a(\nu,r)=\lim_{t\to\infty}\frac{\mathbb{P}_\nu(\tau_\partial>t+r)}{\mathbb{P}_\nu(\tau_\partial>t)}=e^{-K\,r}.$$

Since by assumption $\lambda_0^\kappa>K$, when $\alpha>0$ part (vii) of Lemma 3.3 tells us that $\lambda_0>0$. The strict positivity of λ_0 together with the assumption $\int_0^\infty\gamma(x)^{-1}\,dx=\infty$ allows us to apply part (ii) of Lemma 3.3, to conclude that the speed measure Γ is finite. When $\alpha=0$ and Γ is infinite the same reasoning holds, leading to a contradiction. Therefore we may assume, in any case, that Γ is finite

Therefore Lemma 4.6 shows that for every compactly supported measure ν

$$-\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}_\nu\big(\tau_\partial>t\big)=\lambda_0^\kappa.$$

In the case of escape to infinity equation (45) implies

$$-\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}_\nu(\tau_\partial>t)=\eta_\nu=K\neq\lambda_0^\kappa.$$

Therefore the assumption $F(\nu,\mathbb{R}_+)=0$ cannot be true and thus by Theorem 2.6 we conclude $F(\nu,\mathbb{R}_+)=1$ and $F(\nu,\infty)=0$. Thus X_t converges from every compactly supported initial distribution ν to the quasistationary distribution $\varphi(\lambda_0^\kappa,\cdot)$. □

The above theorem has the following Corollary, which in a slightly more restrictive form already appears in the work [12] of Collet, Martínez and San Martín. The proof presented in [12] suffers from a gap, so it seems to be worth presenting an alternative (and more general) proof of the assertion.

Corollary 4.8. *Suppose $\kappa \equiv 0$ and ∞ is a natural boundary point, and the process X_t is recurrent, with $\alpha > 0$. Then*

- *if $\lambda_0 > 0$ then X_t converges from every compactly supported initial distribution ν to the quasistationary distribution with Γ-density proportional to $\varphi(\lambda_0, \cdot)$;*
- *if $\lambda_0 = 0$ then X_t started from ν escapes to infinity.*

Proof. The first part of the assertion follows directly from Theorem 4.7. In order to prove the second assertion, observe that the function

$$R(y) := \frac{1}{1+\alpha} + \frac{\alpha}{1+\alpha}\int_0^y \gamma(x)^{-1} dx$$

satisfies $LR = 0$; since $R(0) = 1/(1+\alpha)$ and $R'(0) = \alpha/(1+\alpha)$ it is in $\mathcal{D}_\alpha$, and is the unique eigenfunction $\varphi(0, y)$. Since $\lambda_0 = 0$, we have

$$\int_0^\infty \varphi(\lambda_0, y)\,\gamma(y)dy = \frac{1}{1+\alpha}\int_0^\infty \gamma(y)dy + \frac{\alpha}{1+\alpha}\int_0^\infty \gamma(y)\int_0^y \gamma^{-1}(x)\,dx\,dy = \infty,$$

by the assumption that ∞ is a natural boundary. □

4.3. **Low killing at infinity: The transient case.**

Theorem 4.9. *Suppose that ∞ is a natural boundary point and that $\int_0^\infty \gamma(x)^{-1}\,dx < \infty$. If $K < \lambda_0^\kappa$ then X_t escapes to infinity from every initial distribution. The rate of escape is exponential with rate $\lambda_0^\kappa - K$, in the sense that for all $z, x > 0$,*

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}_x\big(X_t \le z \mid \tau_\partial > t\big) = -(\lambda_0^\kappa - K). \tag{48}$$

Proof. Observe that the condition $\int_0^\infty \gamma(x)^{-1}\,dx < \infty$ implies that for each $a \in (0,\infty)$ and each $x \in (a,\infty)$ the unkilled diffusion (corresponding to the generator L) started from x has nonzero probability of never hitting a. For $\varepsilon > 0$ we can choose $a = a_\varepsilon \in (0,\infty)$ such that $\kappa(x) \in (K-\varepsilon, K+\varepsilon)$ for every $x \in [a,\infty)$. Then we have for every $x \in (a,\infty)$

$$\begin{aligned}\mathbb{P}_x\big(\tau_\partial > t\big) &= \mathbb{E}_x\Big[e^{-\int_0^t \kappa(X_s)\,ds},\, T_0 > t\Big]\\ &\ge e^{-(K+\varepsilon)t}\mathbb{P}_x\big(T_a > t\big)\\ &\ge e^{-(K+\varepsilon)t}\mathbb{P}_x\big(T_a = \infty\big)\end{aligned} \tag{49}$$

Since $\mathbb{P}_x\big(T_a = \infty\big)$ is an increasing function of x, we can apply the Markov property to see that there is a nonzero increasing function $C(x)$ such that for all $x > 0$

$$\mathbb{P}_x\big(\tau_\partial > t\big) \ge \mathbb{P}_x\big(X_1 \ge a+1\big)\cdot \inf_{x' \ge a+1} \mathbb{P}_{x'}\big(\tau_\partial > t-1\big) \ge C(x)e^{-(K+\varepsilon)t} \tag{50}$$

On the other hand, for any fixed $z \ge 0$ we can apply the bound (13) and Lemma 2.2 to see that

$$\begin{aligned}\mathbb{P}_x\big(X_t \le z,\, \tau_\partial > t\big) &= (e^{-tL^\kappa}\mathbf{1}_{[0,z]})(x)\\ &\le \big(C_\alpha(x)\lambda_0^\kappa + C'_\alpha\big)\,\|\mathbf{1}_{[0,z]}\|e^{-t\lambda_0^\kappa}\end{aligned} \tag{51}$$

for all $t > 1/2\lambda_0^\kappa$. Combining (50) and (51), we see that there is a constant C' such that

$$\mathbb{P}_x\big(X_t \le z \mid \tau_\partial > t\big) \le \|\mathbf{1}_{[0,z]}\| \frac{C'}{C(x)} e^{-(\lambda_0^\kappa - K - \varepsilon)t}. \tag{52}$$

We conclude that for all $x \ge a_\varepsilon$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big(X_t \le z \,\big|\, \tau_\partial > t\big) \le -(\lambda_0^\kappa - K) + \varepsilon.$$

By Proposition 3.13, since ε is arbitrary, we conclude that the limsup is no more than $-(\lambda_0^\kappa - K)$.

In particular, we have shown that the process escapes to infinity. By Lemma 4.5, it follows that $\lim_{t\to\infty} \mathbb{P}_x\big\{\tau_\partial > t+1 \,\big|\, \tau_\partial > t\big\} = e^{-K}$, from which follows that

$$\lim_{t\to\infty} t^{-1} \log \mathbb{P}_x\big\{\tau_\partial > t\big\} = -K.$$

Lemma 3.14 tells us that

$$\lim_{t\to\infty} t^{-1} \log \mathbb{P}_x\big\{X_t \le z\big\} = -\lambda_0^\kappa,$$

from which (48) follows immediately. □

If κ is eventually constant — that is, for some a we have $\kappa(x) = K$ for all $x \ge a$ — then we can strengthen the conclusion of Theorem 4.9 slightly.

Corollary 4.10. *Suppose that κ is eventually constant and that $\lambda_0^\kappa > 0$. Then for every $x, z \in (0,\infty)$*

$$\sup_t e^{(\lambda_0^\kappa - K)t} \mathbb{P}_x\big(X_t \le z \mid \tau_\partial > t\big) < \infty.$$

Proof. If κ is eventually constant, then (49) and (50) hold with $\varepsilon = 0$, hence (52) as well. □

Remark 4.11. *The case $\kappa \equiv 0$ corresponds to the setting considered in* [35]. *Theorem 4 of* [35] *includes a slightly weaker version of the result in Corollary 4.10, obtained by different methods. The above theorem shows that when $\kappa \equiv 0$ the $\mathfrak{L}^2$-eigenvalue λ_0^κ gives the exponential convergence rate at which X_t escapes to infinity.*

As already mentioned in Remark 2.5 a quasilimiting distribution $\tilde\nu$, which in our case is a probability measure on $(0,\infty)$ is always quasistationary in the sense that for every Borel set $A \subset (0,\infty)$

$$\mathbb{P}_{\tilde\nu}\big(X_t \in A \mid \tau_\partial > t\big) = \tilde\nu(A);$$

but the converse need not hold true. In the cases where we know there is no quasilimiting distribution, though, because the process escapes to ∞, we can show that there is also no quasistationary distribution.

Corollary 4.12. *Let ∞ be a natural boundary, with $\lambda_0^\kappa > K$ and $\int_0^\infty \gamma(x)^{-1}\,dx < \infty$. Then there is no quasistationary distribution.*

Proof. Assume that $\tilde\nu$ is a general quasistationary distribution. The measure $\tilde\nu$ is absolutely continuous with respect to Γ with a positive continuous density $g : [0,\infty) \to (0,\infty)$ (for a sketch of the proof of this fact we refer to the appendix). There is a λ such that $\mathbb{P}_{\tilde\nu}(\tau_\partial > t) = e^{-\lambda t}$. By (50), for any positive ε,

$$e^{-\lambda t} = \mathbb{P}_{\tilde\nu}\big\{\tau_\partial > t\big\} \ge e^{-(K+\varepsilon)t} \int C(x) d\tilde\nu(x), \tag{53}$$

which means that $\lambda \le K$. For any fixed $x_0 > 0$,

$$\begin{aligned} \mathbb{P}_{\tilde\nu}\big\{X_t \le z,\, \tau_\partial > t\big\} &= \big\langle g, e^{-tL^\kappa}\mathbf{1}_{(0,z]}\big\rangle \\ &= \big\langle g\mathbf{1}_{[0,x_0]}, e^{-tL^\kappa}\mathbf{1}_{(0,z]}\big\rangle + \big\langle r\mathbf{1}_{(x_0,\infty)}, e^{-tL^\kappa}\mathbf{1}_{(0,z]}\big\rangle \\ &\le \big\|g\mathbf{1}_{[0,x_0]}\big\| \cdot \big\|e^{-tL^\kappa}\mathbf{1}_{(0,z]}\big\| + \sup_{x\ge x_0}\mathbb{P}_x\big\{X_t \le z,\, \tau_\partial > t\big\} \end{aligned} \tag{54}$$

Since $\|g\mathbf{1}_{[0,x_0]}\|$ and $\|\mathbf{1}_{(0,z]}\|$ are both finite, we can use (5) and (51) to see that there is a constant B such that

$$\mathbb{P}_{\tilde\nu}\big\{X_t \le z,\, \tau_\partial > t\big\} \le Be^{-t\lambda_0^\kappa}. \tag{55}$$

Combining (53) and (55), we see that for all positive t,

$$\tilde\nu\big([0,z]\big) = \mathbb{P}_{\tilde\nu}\big\{X_t \le z \;\mid\; \tau_\partial > t\big\} \le Be^{-(\lambda_0^\kappa - K)t}, \tag{56}$$

so $\tilde\nu$ must be identically 0 on $[0,\infty)$. □

4.4. **Processes that may not hit 0.** Consider a process which is killed only at 0 (that is, with $\kappa \equiv 0$.) If the process is not almost surely absorbed at 0 eventually — that is, if $\mathbb{P}_x(T_0 = \infty) > 0$ — we may wish to condition the process at time t on being killed eventually, but not yet. That is, we consider the long-time asymptotics of

$$\mathbb{P}_x(X_t \in \cdot \mid T_0 \in (t,\infty)).$$

Conditions of this kind can often be found in the analogous problems in the theory of branching processes. This problem can be reduced to our previous analysis by an h-transform. The function $h(x) = \mathbb{P}_x(T_0 < \infty)$ is harmonic, and by general theory (see [40], chapter 4, sections 3 and 10) the process (X_t) conditioned to hit 0 corresponds to the generator L^h whose action is given by

$$L^h f = (\frac{1}{h}L(h\,f))(x) = -\frac{1}{2}f''(x) + \left(-b(x) - \frac{h'(x)}{h(x)}\right)f'(x)$$

The process associated to the operator L^h can again be defined by Dirichlet form techniques, and the associated family of measures on the path space is denoted by $\tilde{\mathbb{P}}_x$. As explained above we have

$$\mathbb{P}_x(\cdot \mid T_0 < \infty) = \tilde{\mathbb{P}}_x(\cdot).$$

The operator L^h can be realized as a self-adjoint operator on the Hilbert space $\mathfrak{L}^2((0,\infty), h(x)^2\gamma(x)dx)$. The transformation $V : \mathfrak{L}^2((0,\infty), h(x)^2\gamma(x)dx) \to \mathfrak{L}^2((0,\infty), \gamma(x)dx)$ defined by $Vf = fh$ is unitary, and defines a unitary equivalence between L and L^h, so the spectrum is invariant under h-transforms. In particular, positivity of the bottom of the spectrum of L implies the positivity of the spectrum of L^h. Since absorption is certain with respect to the measure $\tilde{\mathbb{P}}_x$ we can apply our previous results in order to conclude that for every Borel set $A \subset (0,\infty)$

$$\lim_{t\to\infty}\mathbb{P}_x(X_t \in A \mid T_0 \in (t,\infty)) = \frac{\int_0^\infty \tilde\varphi^h(\lambda_0,x)h(X)\gamma(x)dx}{\int_0^\infty \tilde\varphi^h(\lambda_0,x)h(X)\gamma(x)dx},$$

where $\tilde\varphi^h(\lambda_0,x)$ is the unique solution of $(L^h - \lambda_0)u = 0$, which satisfies $\tilde\varphi^h(\lambda_0,0) = 0$ and $(\tilde\varphi^h)'(\lambda_0,0) = 1$.

Remark 4.13. *It seems to be a rather general principle that there are three possibilities. The first possibility is the non-existence of quasistationary distributions. If there exists a quasistationary distribution then it is either unique or there is a whole continuum of quasi-stationary distributions parameterized by a real interval. This is at least true for birth and death processes on the non-negative integers,* cf. [10].

4.5. **The Case of an Entrance Boundary at ∞.** Observe that $\int_0^\infty \gamma(x)^{-1}dx = \infty$ if ∞ is an entrance boundary. This follows from the fact that in this situation the total speed measure $\int_0^\infty \gamma(x)dx$ must be finite. Thus, the situation is essentially the same as in Theorem 4.7. Indeed, we always have convergence to quasistationarity if ∞ is an entrance boundary.

Theorem 4.14. *Assume that* 0 *is regular and that* ∞ *is an entrance boundary. Then the bottom of the spectrum is an isolated eigenvalue with associated non-negative eigenfunction* $u_{\lambda_0^\kappa}$*. From every compactly supported initial distribution* ν*, the process* X_t *converges to the distribution with density* $u_{\lambda_0^\kappa}/\int_0^\infty u_{\lambda_0^\kappa}(x)\gamma(x)dx$ *with respect to* Γ.

Proof. The first assertion follows from Theorem 3.16. Lemma 4.2 directly implies that X_t converges to the quasistationary distribution $u_{\lambda_0^\kappa}$ from every compactly supported initial distribution if and only if $\int_0^\infty u_{\lambda_0^\kappa}(y)\,\gamma(y)dy$ is finite. Since we are assuming that 0 is regular and ∞ is an entrance boundary the speed measure Γ must be finite. Thus the $\mathfrak{L}^2(\Gamma)$ function $u_{\lambda_0^\kappa}$ is also in $\mathfrak{L}^1(\Gamma)$. □

Remark 4.15. *The uniqueness of quasistationary distributions (assuming their existence) was addressed in the recent paper* [9]*, for the case* $\kappa \equiv 0$*. There it is shown that the uniqueness of quasistationary distributions in this case is equivalent to the assertion that for any* $a > 0$ *there exists* $y_a > 0$ *such that* $\sup_{x>y_a} \mathbb{E}_x\big[e^{aT_{y_a}}\big] < \infty$*, where* T_{y_a} *denotes the first hitting time of* y_a*. Thus uniqueness of quasistationary distributions is equivalent to the 'time of implosion from infinity into the interior' having exponential moments of all orders. It is also proved in* [9] *that this is equivalent to* ∞ *being an entrance boundary. If* $\kappa \equiv 0$*, and if absorption is certain, our results show that the existence of a quasistationary distribution is equivalent to the existence of exponential moments of the first hitting time of* 0*. Both results together account for the existence and uniqueness of quasistationary distributions.*

Appendix

In this Appendix we sketch a proof of the regularity of quasistationary distributions of one-dimensional diffusions with one regular boundary.

Lemma A. *Let* L^κ *be one of the selfadjoint realizations considered in this work, of the Sturm-Liouville expression* $\tau + \kappa$ *in* $\mathfrak{L}^2$*; and let* $\tilde{\nu}$ *be a quasistationary distribution. Then* $\tilde{\nu}$ *is absolutely continuous with respect to the measure* Γ*, with a positive and continuous density* $g : [0, \infty) \to \mathbb{R}$.

Proof. The main assertion of the Lemma will be almost obvious to readers who are familiar with regularity theory for stationary distributions. Indeed, the main strategy we follow is very similar to the case of stationary distributions. Straightforward arguments show that $\tilde{\nu}$ is absolutely continuous with respect to the measure Γ. Denote by g the density of $\tilde{\nu}$ with respect to Γ. The equation

$$e^{-\lambda t}\tilde{\nu}(f) = \mathbb{E}_{\tilde{\nu}}\big[f(X_t); \tau_\partial > t\big];\ \lambda \geq 0,\ f \in C_c((0, \infty)),$$

which results from quasistationarity of $\tilde{\nu}$, implies that

$$\forall f \in C_c^\infty((0,\infty)) : \langle \tilde{\nu}, (L^\kappa + \lambda) f \rangle = \int g(x)(L^\kappa + \lambda) f(x)\, d\Gamma(x) = 0. \tag{57}$$

This means that for any $0 < c < d$,

$$g \in \mathcal{D}(T_{c,d}^*) \text{ and } T_{c,d}^* g = 0,$$

where $T_{c,d}^*$ denotes the adjoint (taken in the Hilbert space $\mathfrak{L}^2((c,d),\Gamma)$) of the minimal operator $T_{c,d}$ defined as the restriction of the differential operator $L^\kappa + \lambda$ to $C_c^\infty((0,\infty))$. The domain of $T_{c,d}^*$ is given by

$$\begin{aligned} \mathcal{D}(T_{c,d}^*) = \big\{ f \in \mathfrak{L}^2((c,d),\Gamma) \,|\, f, \gamma f' &\text{ absolutely continuous in } (c,d) \text{ and} \\ &\frac{-1}{2\gamma}(\gamma f')' + (\kappa+\lambda) f \in \mathfrak{L}^2((c,d),\Gamma) \big\} \end{aligned} \tag{58}$$

and for $f \in \mathcal{D}(T_{c,d}^*)$ one has $T_{c,d}^* = \frac{-1}{2\gamma}(\gamma f')' + (\kappa + \lambda) f$. Since $c, d \in (0,\infty)$ are arbitrary we conclude that

$$\frac{-1}{2\gamma}(\gamma f)'(x) + (\kappa+\lambda) f(x) = 0 \text{ in } (0,\infty) \tag{59}$$

Due to the regularity of the boundary point 0 we conclude (using standard ODE theory) that

$$\lim_{x \to 0+} r(x) \in \mathbb{R}$$

exists. □

Acknowledgements

The authors would like to thank Ross Pinsky for sending them his work [41] prior to publication and for his useful remarks. Moreover we would like to thank Gady Kozma for sending us his work concerning the Davies' conjecture. The hospitality of Worcester College, Oxford for M.K. in support of this collaboration is gratefully acknowledged.

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